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Technical Report

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ABSTRACT

The dynamic response of Lake Warner (Hadley, Massachusetts) to pollution abatement was studied by field measurements and mathematical modeling. Stream flow and phosphorus concentration measurements were used to describe phosphorus flow through the lake system. The lake system released phosphorus, presumably from the sediments, during July 1981, but retained phosphorus during the fall of 1981. Seepage through lake sediments was measured using seepage meters and was determined to be insignificant in the water and phosphorus budget of Lake Warner.

Differential equations for the lake water, sediment interstitial water and sediment solid phase phosphorus concentration developed by others in previous studies were solved using a trial function method. Approximate solutions were developed using steady state and pseudo-order assumptions. The analytic solution was programmed on a hand-held calculator. Sensitivity analysis was carried out for model parameters. The analytic expression for the solution to the differential equations describing lake recovery was an important tool for lake planning, management and research purposes.

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NOMENCLATURE

Symbol	Dimensions in Mass (M), Length (L) and Time (T).
A	Area (L^2), square meters
C	Lorenzen (1973), Average Annual Nutrient Concentration (M/L^3), grams per cubic meter
C_s	Lorenzen (1973), Nutrient Concentration in Sediment (M/L^3), grams per cubic meter
Dr	Sediment Reactive Depth (L), meters
K_1	Release Rate (L/T), meters per day
K_2	Sedimentation Rate (1/T), per day
K_3	Conversion Rate (1/T), per day
k	Lorenzen (1973), Net Specific Rate of Loss to Sediment (M/T), meters per year
k_1	Lorenzen (1973), Specific Rate of Nutrient Transfer to Sediments (M/T), meters per year
k_2	Lorenzen (1973), Specific Rate of Nutrient Transfer from Sediment (M/T), meters per year
M	Mass Flow In from all Sources (M/T), grams per year
P_i	Sediment Interstitial Total Phosphorus Concentration (M/L^3), micrograms per liter
P_{io}	Sediment Interstitial Phosphorus Concentration Initial Condition (M/L^3), micrograms per liter
$P_{i\infty}$	Sediment Interstitial Phosphorus Concentration, Steady State (M/L^3), micrograms per liter
P_L	Lake Water Total Phosphorus Concentration (M/L^3), micrograms per liter
P_{Lo}	Lake Water Total Phosphorus Concentration Initial Condition (M/L^3), micrograms per liter
$P_{L\infty}$	Lake Water Total Phosphorus Concentration, Steady State (M/L^3), micrograms per liter

P_s	Sediment Solid Phase Phosphorus Concentration (M/L^3), micrograms per liter
P_{so}	Sediment Solid Phase Phosphorus Concentration, Initial Condition (M/L^3), micrograms per liter
$P_{s\infty}$	Sediment Solid Phase Phosphorus Concentration, Steady State (M/L^3), micrograms per liter
Q	Average Annual Flow Rate (L^3/T), cubic meters per day
R_{rel}	Rate of Release from Lake Bottom (M/L^3-T), micrograms per liter per day
R_{sed}	Rate of Sedimentation to Lake Bottom (M/L^3-T), micrograms per liter per day
r	Root of cubic equation ($1/T$), per day
T	Hydraulic residence time (T), days
u	Macro-Seepage Velocity (L/T), micrometers per second
V	Volume (L^3), liters
V_L	Lake Water Volume (L^3), cubic meters
V_s	Volume of Sediment (L^3) cubic meters
$X_1 \ldots X_7$	Lake Model Parameters ($1/T$), per day
X_8	Lake Model Parameter (M/L^3-T), micrograms per liter per day
Z	Lake Mean Depth (L), meters
ε	Sediment Porosity
α, β and γ	Lake Model Parameters ($1/T$), per day

CHAPTER I

INTRODUCTION

The eutrophication of many freshwater lakes has been accelerated by cultural activities such as wastewater overflows and nonpoint pollution. Lake restoration, funded under Section 314 of the Federal Water Pollution Control Act Amendments of 1972, includes the diversion of wastewater overflows. After such diversions or other water pollution control measures, lake sediments can act as a nutrient source for biological activity and, therefore, control the rate of recovery of a eutrophic lake.

Lake nutrient budget models can be used by planners and engineers to predict a lake's response to pollution control efforts. When constructing a nutrient budget for a lake one must account for all nutrient sources, such as lake sediments and ground water discharge/recharge, which may effect the recovery of a eutrophic lake.

The recovery of Lake Warner (Massachusetts) was the subject of past studies by Snow and DiGiano (1976) and Bingham and Feng (1980). Wastewater overflows were diverted from Lake Warner in 1975, thus making it a convenient site for testing the lake recovery model developed by Snow and DiGiano (1976). This model predicted that it would take about 14 years for Lake Warner to recover from its eutrophic state. Subsequent field work by Bingham and Feng (1980) showed that Lake Warner had recovered in just one year. Hence, they modified the Snow and DiGiano model to include a reaction rate (K_3) which describes the conversion of sediment solid phase phosphorus to sediment interstitial phosphorus. The value of this reaction rate may control the recovery of a eutrophic lake when sediments contain a significant reservoir of phosphorus.

Objectives

The two major goals of this research were to examine the significance of ground water-lake interactions and to seek an analytical solution to the ordinary differential equations describing phosphorus in the lake recovery model. The hypothesis that ground water may act as a transport mechanism for recycling lake sediment nutrients into the lake water was tested by measuring the ground water discharge/recharge through Lake Warner sediments using seepage meters. It was also of interest to ascertain lake sediment phosphorus release through a mass balance on lake water phosphorus.

An analytical solution to the lake recovery model was desired to eliminate the need for a high speed computer used to implement the numerical solutions of past studies. Simplifications of the governing differential equations using approximation techniques were studied in an effort to produce a compact model for use in lake planning, management and research purposes. A simplified model was deemed acceptable if predictions closely matched the results of the full analytical solution.

Sensitivity analysis was carried out on model parameters to determine the effect variations in the values used have on model predictions. This information can be used to show which parameters must be estimated very accurately and which parameters can be approximated without significantly affecting model predictions.

Scope

Many lakes in Massachusetts are similar to Lake Warner in that they are relatively shallow, have short hydraulic residence times and can be treated as being completely mixed for time periods greater than a few hydraulic residence times. Many of these lakes were formed by man diking a stream. This study is concerned with the phosphorus budget in this category of lakes. Lake Warner will be used as an illustrative case study. Quabbin Reservoir illustrates a man-made reservoir whose phosphorus budget is unlikely to be closely described by the methods illustrated in this study.

The role of ground water in affecting the phosphorus budget of a lake has not been clearly illustrated in past studies of Lake Warner. This study reviews literature on observations of ground water-lake interactions and uses field measurements for assessing the importance of ground water to Lake Warner's water and phosphorus budgets. Field measurements on the Mill River phosphorus loading and on the lake water are used to evaluate the release of phosphorus from Lake Warner sediments. An analytical solution to the Bingham and Feng (1980) mathematical model of the recovery of a eutrophic lake is presented. Equations for the lake water phosphorus, sediment interstitial phosphorus and sediment solid phase phosphorus are presented. These equations provide tools which engineers and scientists can use to determine the values of reaction rates which govern phosphorus cycling in shallow lake systems.

Sensitivity analysis is used to show the effects on model predictions due to marginal variations in input parameters.

CHAPTER II

LITERATURE REVIEW

This section reviews literature on phosphorus chemistry in lake environments. Lake phosphorus budget modeling and mathematical methods for solving the resulting equations are examined. The importance of groundwater seepage in the water and nutrient budgets of lakes is discussed.

Phosphorus Chemistry

Carbon, nitrogen and phosphorus are required by plant life in ratios of approximately 100:15:1, respectively. Phosphorus has no gaseous phase and is relatively scarce, therefore making it a limiting nutrient to plant growth in most natural aquatic ecosystems. Phosphorus in nature generally occurs as phosphate in both dissolved and solid phases. Categories of phosphate can be operationally defined by particle size and acid molybdate reactivity.

Because of complex limnological transformations in lake systems (see Figure 1) dissolved phosphate concentrations are not a good index of the biologically available phosphorus during the growing season. Schindler et al. (1971) claim that the amount of phosphorus supplied rather than the amount in solution at any

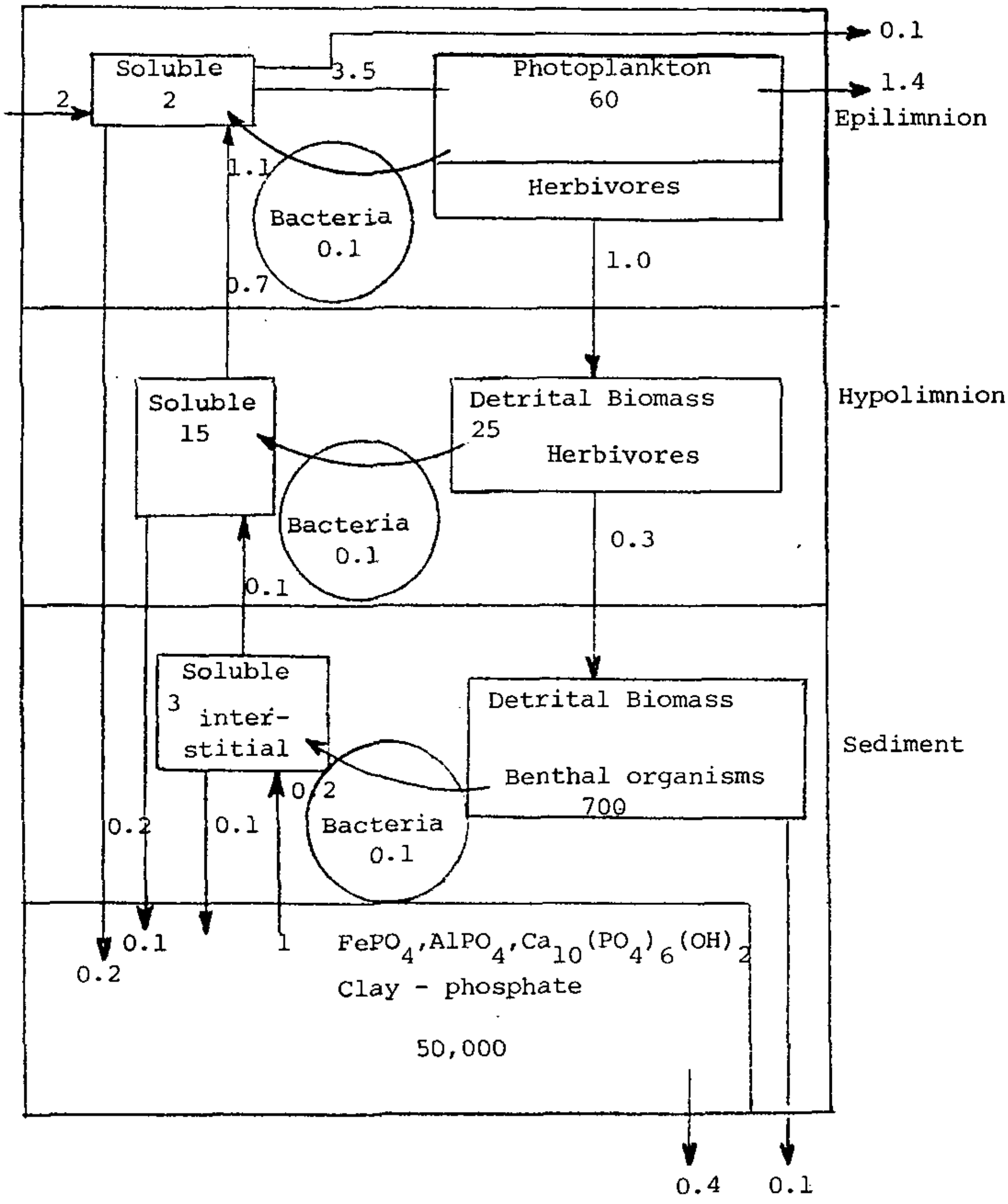


Figure 1. Transformations of phosphorus in a hypothetical lake system. This steady state model shows representative phosphorus concentrations for model compartments (μg/l) and numbers on arrows show mass flow rates (μg/l-day) (Stumm and Leckie, 1971).

point in time must be known to predict algal growth. Lee et al. (1978) note that for most water bodies the biologically available phosphorus is between the total phosphorus and soluble orthophosphate loading since some of the particulate phosphorus is solubilized (Stumm and Morgan, 1981).

The concentration of dissolved phosphate in natural waters gives little indication of phosphate availability for biological growth. The residence time of dissolved phosphate in natural waters ranges from 0.0021 to 8.3 days (Stumm and Morgan, 1981). Lean (1973) found that fifty percent of $^{32}PO_4$ was taken up by algae in less than two minutes. Schindler et al. (1971) found that dissolved phosphate concentrations were undetectable within an hour after artificially raising the phosphate concentration of a lake to 10 g/l. Dissolved inorganic phosphate concentrations in water fluctuate on a daily cycle (Vollenweider, 1968). Low dissolved phosphate concentrations were found at night and higher phosphate concentrations were found during the day. It was suggested that phosphates were required by phytoplankton for energy conversion at night and that some phosphates were released by growing cells during the day.

Lake bottom phosphorus occurs in the sediment solid phase and in the sediment interstitial water. Phosphate adsorption by lake sediments is governed by sediment composition and by physical and chemical conditions in the sediment such as redox potential, pH and temperature. Under anoxic lake bottom conditions (which might occur during summer stagnation periods) a lower redox potential in the sediment might develop causing an increase in the dissolution of Fe bound phosphorus (Ku et al., 1978). Also the phosphorus binding capacity of sediments is greater as pH decreases. Figure 2 shows the effect of pH on Fe bound phosphate. It is evident that the dissolution of sediment phosphorus would increase in response to high pH levels which would occur during summer algal blooms.

Many researchers believe that phosphorus release from lake bottoms is from sediment interstitial phosphorus (Syers et al., 1973). Hence the dissolution of sediment solid phase phosphorus to sediment interstitial phosphorus may govern the release of phosphorus to the lake water.

Phosphorus released as orthophosphate from lake bottoms undergoes rapid transformation into other more stable compounds which are components of total phosphorus. Hence, when measuring release rates (such as Snow and DiGiano's K_1) total phosphorus is used as a measure of the phosphorus released from lake bottoms.

Lake Phosphorus Budget Modeling

Lakes which are phosphorus limited exhibit good correlations between total phosphorus concentration and primary productivity parameters such as chlorophyll a (Williams et al., 1978; Dillon and Rigler, 1974; Jones and Bachmann, 1976).

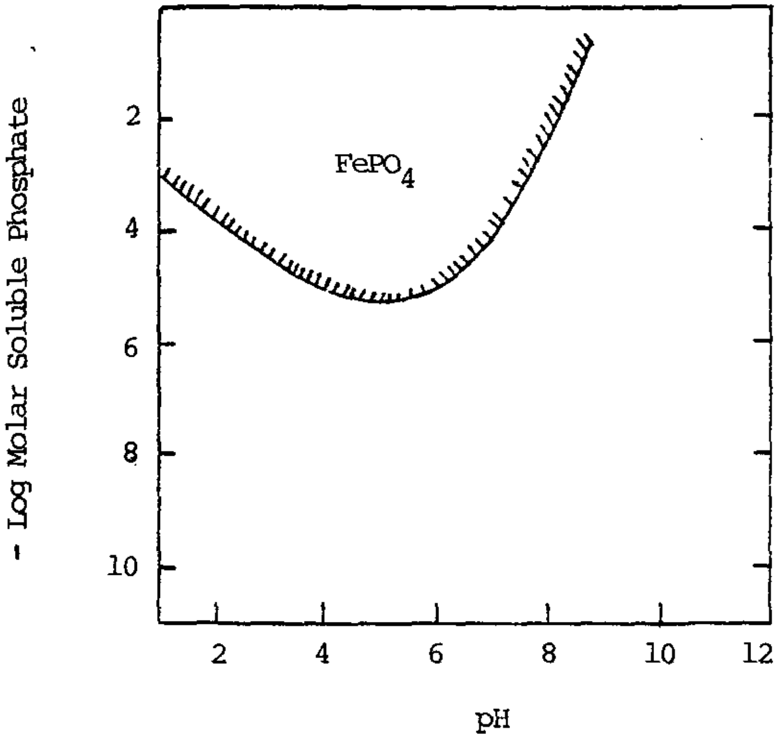


Figure 2. Solubility diagram for iron phosphate (after Stumm and Leckie, 1971).

Phosphate sources to aquatic systems result from natural weathering and dissolution of phosphate minerals, soil erosion, soil fertilization, biological transfer, detergents and domestic and industrial wastewaters (Stumm and Morgan, 1981). Table 1 shows phosphorus export coefficients based on the U.S. Organization for Economic Cooperation and Development (OECD) eutrophication study.

Vollenweider (1968) developed an empirical model which relates total phosphorus loading to a lake's trophic state. When areal phosphorus loading (L(P)) was plotted against mean depth (Z) on a log-log plot (figure 3), lakes appear to be grouped according to their trophic states. Lines dividing trophic states define critical phosphorus loading boundaries. A line dividing the oligotrophic and eutrophic states has the equation

$$L_c(P)\ (\mathrm{mg/m^2\text{-}yr}) = (25 \text{ to } 50)Z^{0.6} \qquad 2.1$$

Dillon (1975) noted that Cameron Lake appeared in the eutrophic zone in this type of plot but did not exhibit the normal characteristics of eutrophy due to a short hydraulic residence time. Vollenweider (1976) refined his criterion axis to include the hydraulic residence time (T). The resulting log-log plot (figure 4) of areal phosphorus loading vs Z/T gives a more widely applicable model of lake trophic status. The line dividing oligotrophic from eutrophic lakes has the equation:

Table 1. Representative watershed total phosphorus export coefficients from the U.S. OECD eutrophication study (after Lee et al., 1978).

Land use	Total Phosphorus ($g/m^2/yr$)
Urban	0.1
Rural/Agruculture	0.05
Forest	0.01
Other:	
rainfall	0.02
dry fallout	0.08

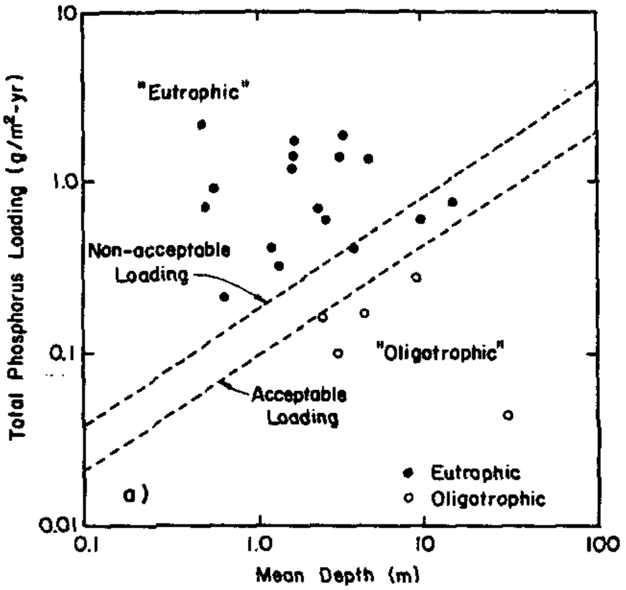


Figure 3. Vollenweider (1968) phosphorus loading - trophic state criterion axes.

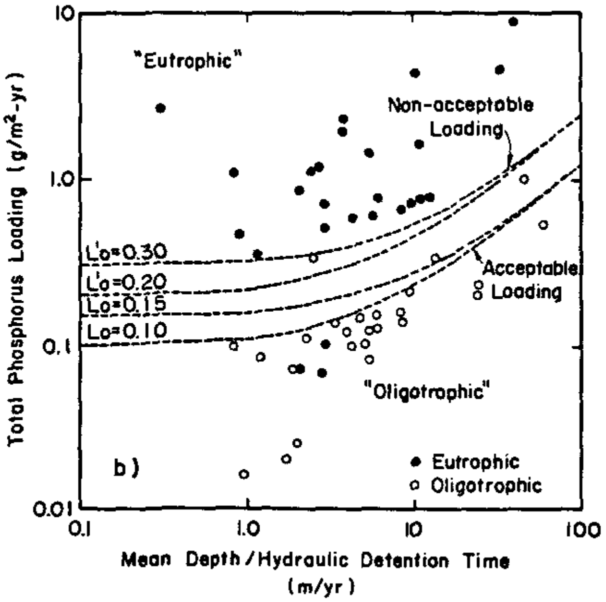


Figure 4. Vollenweider (1976) phosphorus loading - trophic state criterion axes.

$$L_c(P) = 100\ (Z/T)^{0.5} \qquad 2.2$$

Vollenweider's phosphorus loading-trophic state criterion axes are useful for establishing maximum allowable phosphorus loadings to lakes. More complex mass balance models of phosphorus dynamics in lake systems are needed to predict the time dependent response of lake phosphorus concentrations to pollution control measures.

Lorenzen (1973) summarizes model derivations which describe the fate of a conservative substance (total phosphorus is the nutrient of concern) in a well mixed lake. Model I (schematically shown in Figure 5) allows for inflow, outflow and sedimentation of phosphorus. Continuous Flow Stirred Tank Reactor (CFSTR) assumptions allow the lake to be treated as completely mixed. Model II (Figure 5) considers the release of phosphorus from the lake bottom. It is assumed that the concentration of phosphorus in the lake bottom does not change over time. Model III (Figure 5) allows for depletion of phosphorus in the lake bottom over time.

Snow and DiGiano (1976) developed a lake nutrient budget model which differs from the Lorenzen Model III in that lake bottom phosphorus release is from the sediment interstitial water. A linear relationship was developed empirically which described the equilibrium concentration of sediment solid phase phosphorus and sediment interstitial phosphorus. This

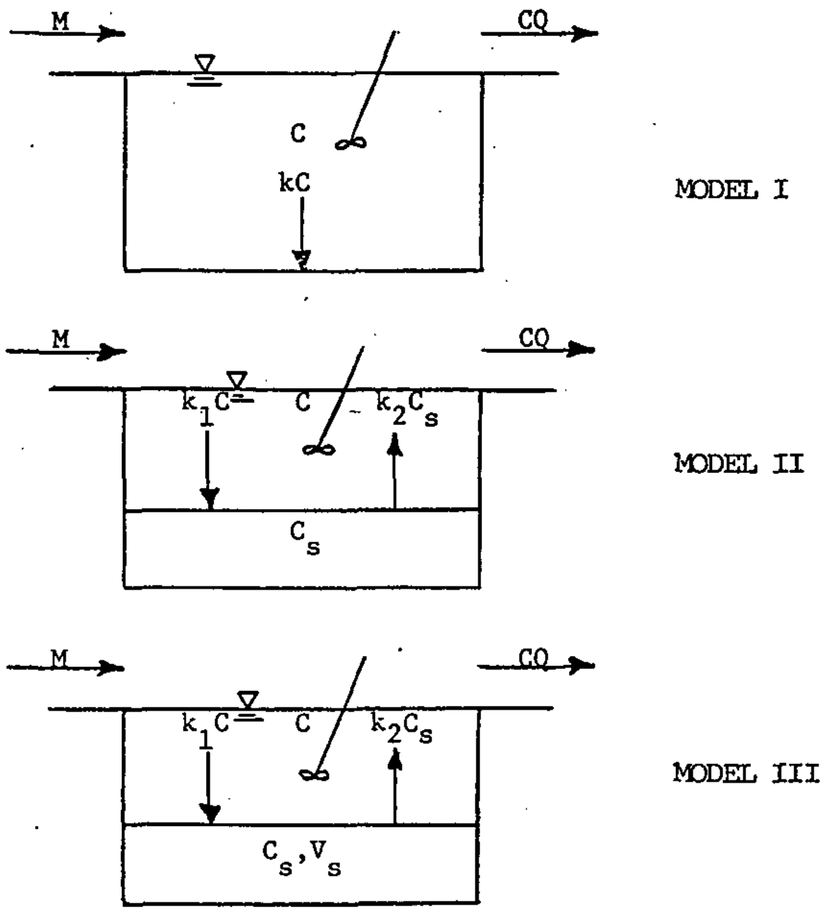


Figure 5. Schematic representation of mass balance models derived by Lorenzen (1973).

assumption, in effect, allowed an instantaneous conversion rate between sediment solid phase and sediment interstitial phosphorus. The Snow and DiGiano model predicted that it would take Lake Warner about 14 years to recover from its eutrophic state in response to a reduction in phosphorus loading. Bingham and Feng (1980) found through field observations that Lake Warner had recovered from its eutrophic state (or reached equilibrium with the reduced phosphorus loading rate) one year after the diversion of wastewater overflows. Hence, they modified the Snow and DiGiano model to include a rate constant which describes the conversion from sediment solid phase to sediment interstitial phosphorus. There is no experimental evidence for choosing a particular value for this rate constant and it is found by model calibration.

Mathematical Methods

The most complex Lorenzen mass balance model (Model III) consists of two simultaneous ordinary differential equations with constant coefficiencies. The equations can be solved numerically or by analytical methods (Lorenzen, 1976). The solution of three simultaneous linear first order ordinary differential equations with constant coefficients is required by the modified Snow and DiGiano model and will be considered in the following discussion on mathematical methods. Methods of solving the equations include the Laplace transform method, the eigenvalue method, the power series method, the trial function method, numerical methods and other methods.

Laplace transform methods are convenient to use for solving the type of equations under consideration here. However, inverse transforms do not always exist (Churchill, 1958) and the solution of more than two simultaneous equations becomes formidable by this method. Similarly, eignevalue methods (Kaplan, 1957) become complex when the solution of a system of three or more equations is attempted.

Manipulations of the equations can be aided by introducing matrix notation and solving the differential equations by numerical or power series methods (Himmelblau et al., 1968). The power series method (Rinaldi et al., 1979) is useful if the series converges quickly.

The system of equations can be solved by first reducing them to a single equation of a higher order (Elsgolts, 1973). The solution of the higher order differential equation may be easier than the direct solution of the system of differential equations.

Bingham and Feng (1980) solved the lake model equations by numerical integration. The solution was carried out in one day time steps on a high speed computer.

A trial function method (Stockton, personal communication; Kaplan, 1957) is practical for solving a maximum of three equations. A solution is postulated and unknown constants are found by manipulating algebraic equations. See Appendix B for details of the trial function method.

Order of magnitude analysis can be used to simplify systems of differential equations by focusing attention on the dominant terms and neglecting small terms. The method is most useful when the simplified equations are solvable whereas the original equations are intractable. One of the most widely recognized applications of the method was Prandtl's 1904 development of boundary layer theory to describe fluid motion near a solid surface (Lamb, 1932).

Approximation methods have been extensively applied in the study of chemical reaction kinetics. The pseudo-order approximation method (Freifelder, 1982), combines experimental planning with mathematical simplication to reduce the order of the differential equation describing a reaction. As rates of formation or consumption of reactants are proportional to the product of the concentration of the reactants raised to a power, the chemist is motivated to combine experimental and mathematical wizardry to transform intractable differential equations into those more amenable to solution. The method of pseudo-order approximation accomplishes the reduction of the reaction rate order by treating a reactant which is in large excess as constant.

Another approximation method resembling order of magnitude analysis is used to simplify the mathematical description of certain consecutive chemical reactions. Consecutive chemical reactions occur when the product of one reaction is the reactant in a subsequent reaction. The method referred to as the stationary state approximation (Harris, 1966), is applied when consecutive reactions occur at different rates, sometimes allowing a slowly changing time derivitive term, describing the rate of formation or consumption of a chemical species, to be set equal to zero. As a result, an unsteady state equation is approximated by an equilibrium or steady state equation. Appropriately, the quasi-steady state method is known in the Russian literature as "the method of quasistationary concentrations" (Emanuel' and Knorre, 1973).

Ground Water-Lake Interactions

Ground water-lake interactions are poorly understood and are often overlooked in lake water and nutrient budgets (Winter, 1978). Some recent studies have used seepage meters to measure, *in situ*, the direction and magnitude of seepage flux through lake sediments. The chemical composition of samples taken from seepage meters may be effected by the environment created by these devices.

Karauskas and Anderson (1978) suggest that flow through conditions occur at Nepco Lake (Wisconsin) based on water table contours around the lake. High manganese concentrations down gradient of Nepco Lake were attributed to the dissolution of manganese in the lake sediments. A similar analysis was done by Loeb and Goldman (1979) to estimate the ground water contribution of Ward Valley to Lake Tahoe. Ground water flow during 1975 was found to be 16% of the Ward Creek inflow to Lake Tahoe. Ground water contributions of nitrates and soluble phosphorus were found to be 78% and 90%, respectively, of the Ward Creek loading.

Dilutional pumping was carried out in 1970 to improve the water quality of Snake Lake (Born et al., 1973). With ground water phosphorus concentrations of about 0.050 mg/l it was expected that the lake water phosphorus concentration (0.4 mg/l) would be diluted. Pumping increased ground water velocities from about 3.5 to 28 μm/sec. The initial increases of chloride, nitrogen and phosphorus in the lake water were thought to be caused by flushing from lake sediments due to the high sediment interstitial velocities induced by pumping.

Cartwright et al. (1979) suggest that ground water movement into Lake Michigan explains the distribution of trace elements in bottom sediments. Trace elements decrease in concentration downward from the lake water--sediment interface. Results from piezometer measurements showed that the ground water flux was from the aquifer into the lake. Hence, the ground water which has twice the mineral content and hardness as the lake water moves through the lake sediments and trace elements are precipitated out near the sediment--lake water interface.

Seepage Meters

The direct measurement of aquifer discharge and recharge in lake environments can be made with seepage meters described by Lee (1977) and by Lock and John (1978). Lee (1977) describes a simple device consisting of a 55 gallon drum with one open end and an outlet on the other end for sample collection. Lock and John (1978) describe a seepage meter provided with a container to buffer the sampling vessel from outside water movement. Also, a gas/pressure release valve was mounted on the top of the meter to allow the measurement of gas which may be released from the sediments during the sampling period.

Theoretical models of ground water-lake interactions can be verified using seepage meters. Also chemical analysis of seepage water may provide information on the effect of sediment chemistry on the chemistry of ground water and vice versa.

Patterns of ground water flow through lake sediments were measured using seepage meters (Fellows and Brezonick, 1980; Connor and Belanger, 1981; Lock and John, 1978; Lee *et al.*, 1980;

Brock et al., 1982). Maximum seepage velocities were found to occur near shore and to decrease exponentially with distance from the shore line.

Lee (1976) examined the reliability of seepage meters as ground water collection devices at Lake Sallie (Minnesota). Nutrient concentrations in seepage water became similar to nutrient concentrations in adjacent wells after more than 1,200 liters of water had passed through the seepage meters. Keel (1979) found that ground water nutrient contributions to lakes in the Upper Pelican River Watershed of Minnesota were significant. The average total phosphorus and total nitrogen concentration of seepage water ranged from 1.04-1.63 and 1.59-4.0 mg/l, respectively.

A theoretical model based on ground water contours around Perch Lake (Ontario) was confirmed using seepage meters and a tritium tracer (Frape and Patterson, 1981). The authors suggest that lake water mixing with sediments would be minimal in aquifer discharge areas which generally occur near shore and deeper mixing would be found in deep lake sediments. Accordingly, it was noted that a thick metal zone occurred in deep lake sediments, due to mixing with oxygen rich lake water, while a thin metal enriched zone was found in near shore sediments. Lee et al. (1980) suggest that sediment biology may affect the ground water chemistry as ground water enters a lake and that ground water may flush materials from lake sediments into the lake.

Brock et al. (1982) estimate ground water seepage into Lake Mendota using seepage meters. Because seepage water has a long residence time in the seepage meters, compared with the time it would take for the seepage water to undergo chemical modification, seepage meter samples were not considered reliable for estimating nutrient flux from ground water. Therefore, both dialysis chambers and pipeting were used to gather sediment pore water samples for chemical analysis.

Spacial variability in the chemical composition of sediment pore water was attributed to either differences in groundwater chemistry or the modification of seepage water by lake sediments as the seepage water moved into the lake. The average phosphorus concentration of seepage water was found to be 171.5 μg/l. Seepage meter data show that ground water accounts for one third of the inflowing water to Lake Mendota. Chemical analysis of seepage water shows, however, that seepage represents only 12% of the phosphorus loading to the lake.

Comparisons of seepage water nutrient concentrations with that of wells in the vicinity indicates that seepage water is higher in phosphorus and ammonia than well water. The authors suggest that the phosphorus and ammonia in seepage water was derived from the lake itself and is, therefore, being recycled from the sediments aided by ground water discharge through the lake sediments.

CHAPTER III

FIELD MEASUREMENTS AND DATA COLLECTION

Background

Lake Warner was created in the early 1800's by the construction of a dam on the Mill River (Hadley, Massachusetts). By-passes from the Amherst Wastewater Treatment Plant flowed into the Mill River, approximately 1.5 miles (2.4 $(10)^3$ m) upstream from Lake Warner, in the late 1960's and early 1970's, causing the lake to experience excessive algal blooms. Wastewater discharges were stopped in November, 1975, with the completion of the new Amherst Wastewater Treatment Plant. Hence, Lake Warner provided a good study area for assessing the effect of reducing nutrient loadings to a eutrophic lake.

The Lake Warner watershed was analyzed by Jubinville (1973). The total area drained by the Mill River was determined to be 32.1 square miles (8313.8 hectares) and the land usage in this watershed is shown in Table 2.

Table 2. Percentage of Land Usage in the Lake Warner Drainage Basin (after Bingham and Feng, 1980)

Land Use Type	Percentage
Agricultural (Dairy Farms)	20%
Agricultural (Crops)	20%
Non-Agricultural (Forest)	50%
Urban	10%

Snow and DiGiano (1976) determined the lake area to be 68 acres (27.5 hectares), the lake volume to be 352 acre-feet (4.35 $(10)^5$ m^3) and the average depth to be 5.15 feet (1.58 m).

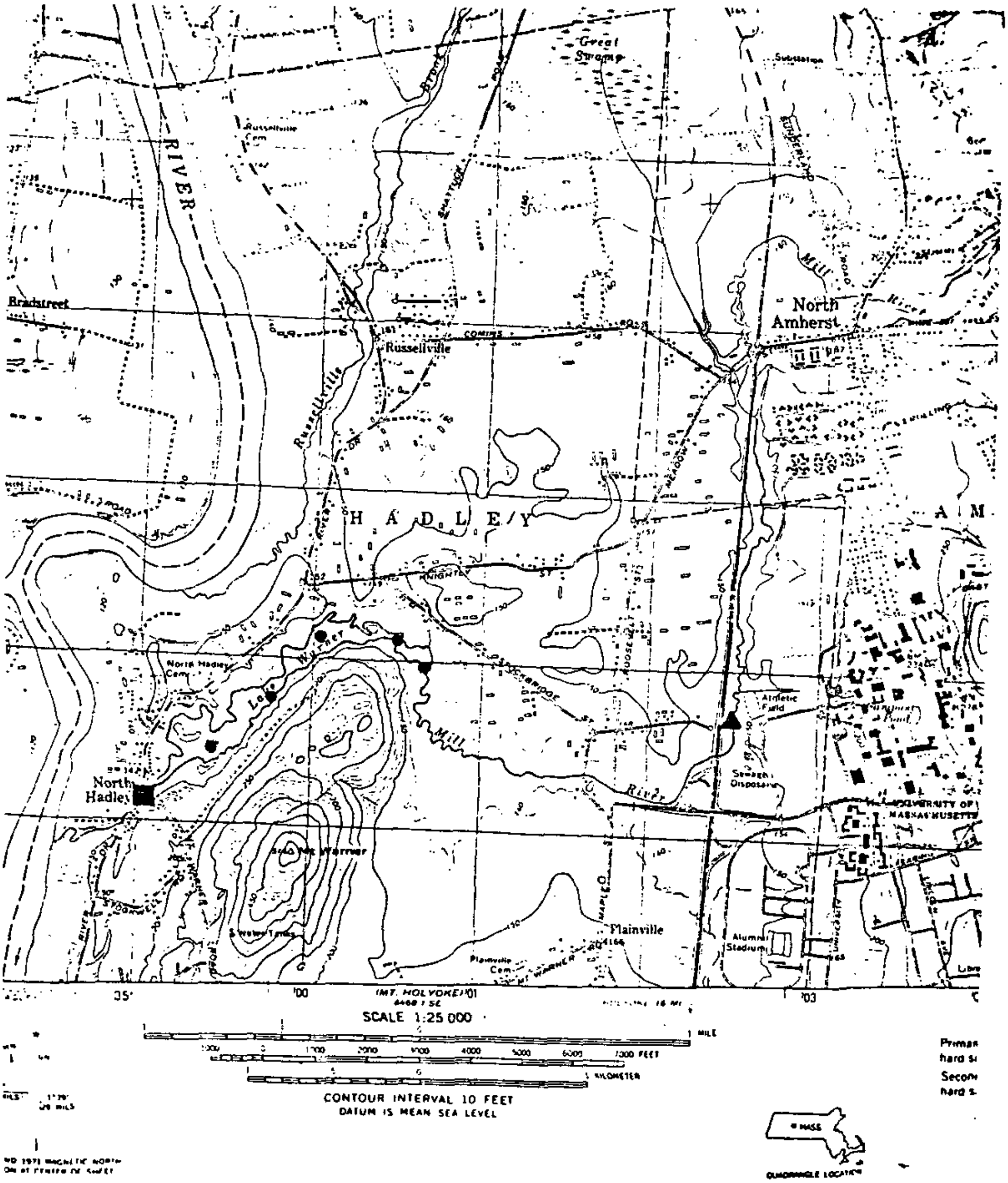
Lake Warner sediments were shown to reflect eutrophic lake characteristics (Snow and DiGiano, 1973) based on the high concentration of PO_4^{-3}-P in the sediment which would be available for exchange with the lake water to support algal blooms. Snow and DiGiano (1976) estimated the average porosity of the upper five centimeters of the lake sediment to be 0.84. Also, high sediment solid phase and interstitial phosphorus concentrations were found in Lake Warner sediments. They measured sediment solid phase phosphorus concentrations of 0.9 to 1.6 mgP/g of dry sediment and 320 to 450 μg/l sediment interstitial total phosphorus concentrations. The lake water total phosphorus concentrations averaged about 90 μg/l in 1973 and 1974. Snow and DiGiano (1976) also found an extreme dissolved oxygen gradient in the lake water during the summer of 1973. Excessive algal blooms (including a blue-green variety) occurred throughout the summers of 1973 and 1974.

Bingham and Feng (1980) found that Lake Warner had recovered from its eutrophic state more quickly than had been predicted by the Snow and DiGiano (1976) lake recovery model. The average total phosphorus concentration of the lake water was found to be 50 μg/l in 1976. Algal blooms were of only very short duration during mid July of 1976. They found that sediment interstitial total phosphorus concentrations ranged from 150 to 300 μg/l and that the sediment solid phase phosphorus concentration was actually higher than that measured in 1973 by Snow and DiGiano (1976). This was attributed to more consistently oxic lake bottom waters which increased the phosphorus binding capacity of the lake sediments.

Visual observations during the summer of 1981 indicate that Lake Warner has indeed recovered from its former eutrophic state. During the low flow and high temperature period of July algal blooms were of only short duration. Filamentous blue-green species were not excessive but Watermeal (Wolffia brasiliensis) and Duckweed (Lemna minor) did cover much of the lake surface during portions of the summer of 1981.

Materials and Methods

To follow up the studies by Snow and DiGiano (1976) and Bingham and Feng (1980) the Mill River and Lake Warner were sampled for phosphorus concentration on July 31, 1981 and daily during a period from September 18, 1981 to October 4, 1981. Stream gauging and sampling was carried out on the Mill River (stations shown in figure 6) to estimate the phosphorus budget of Lake Warner and to model the release of phosphorus from the lake bottom. Also, measurements were made on the ground water component of Lake Warner hydrology using seepage meters. Measurements taken during the dry months of July and August are compared to



Symbol	Sampling Location
▲	Influent sampling location for mass balance shown in Figure 11.
■	Effluent sampling location used in Figure 11.
●	Sampling locations used in Figures 9 and 10.

Figure 6. Location map and water sampling sites.

measurements taken during October and November of 1981 (see figure 13).

Water samples were analyzed for total and orthophosphate phosphorus using the Heteropoly Blue-Ascorbic acid Spectrophotometric method (Strickland and Parsons, 1972). Samples from Lake Warner were taken at one-third meter below the surface and from one meter above the bottom. Bottom samples were taken with a Kemmerer sampler. Samples were transported to the laboratory in 500 milliliter acid washed bottles. All glassware used in the phosphorus analysis were acid washed.

Water samples were analyzed for both orthophosphate and total phosphorus on the day that they were collected. Samples analyzed for orthophosphate were filtered through 0.45 m filters, using a millipore apparatus, prior to the addition of the composite reagent solution. Total phosphorus samples were autoclaved after the addition of 0.4 ml of 11N H_2SO_4 and 0.2 grams of potassium persulfate. These samples were allowed to cool to room temperature before pH adjustment and the addition of the composite reagent solution. The absorbance of the prepared samples was measured on a Bausch and Lomb Spectronic 70 spectrophotometer. A ten centimeter cell was used. The calibration curves of total and orthophosphate phosphorus concentration vs. absorbance appear in Appendix A.

Stream gauging was done on the inlet and outlet streams of Lake Warner using the U.S.G.S. mid-section method (Buchanan and Somers, 1969). Stream velocities were measured at 0.6 of the stream depth with a Gurley pygmy current meter. A stage-discharge curve was developed using stream flow measurements and a staff gauge to measure stream stage. These curves appear in Appendix A. Stream flow on the sampling dates was estimated by measuring the stream stage and finding the flow from the stage-discharge curve.

Seepage through Lake Warner sediments was measured during July, August and November 1981 using seepage meters described by Lee (1977). The basic shell of aseepage meter consists of a 208.2 liter (55 gallon) steel drum cut at 25 centimeters (9.8 in.) from the end (see Fig. 7). The insides of the seepage meters were painted with epoxy resin to prevent corrosion. The seepage meters were installed in the lake bottom by turning them slowly into the lake sediments until approximately ten centimeters of the meter extended above the sediment-lake water interface. Tubes for sample collection were then inserted in the the bung holes which had been cut into the tops of the meter and Ziplock heavy duty freezer bags were attached to them. Five hundred milliliters (0.132 gal.) of distilled water was placed in each bag before it was attached to a seepage meter.

Meters 1 through 5 were installed on the southeast side of Lake Warner and meters 1A through 5A were installed on the opposite side of the lake as can be seen from Figure 8. The meters were placed in the lake sediments 1.5 to 3 meters (five to ten feet) from shore in about 0.6 meters (two feet) of water. Sample bags were changed from a twelve foot aluminum boat.

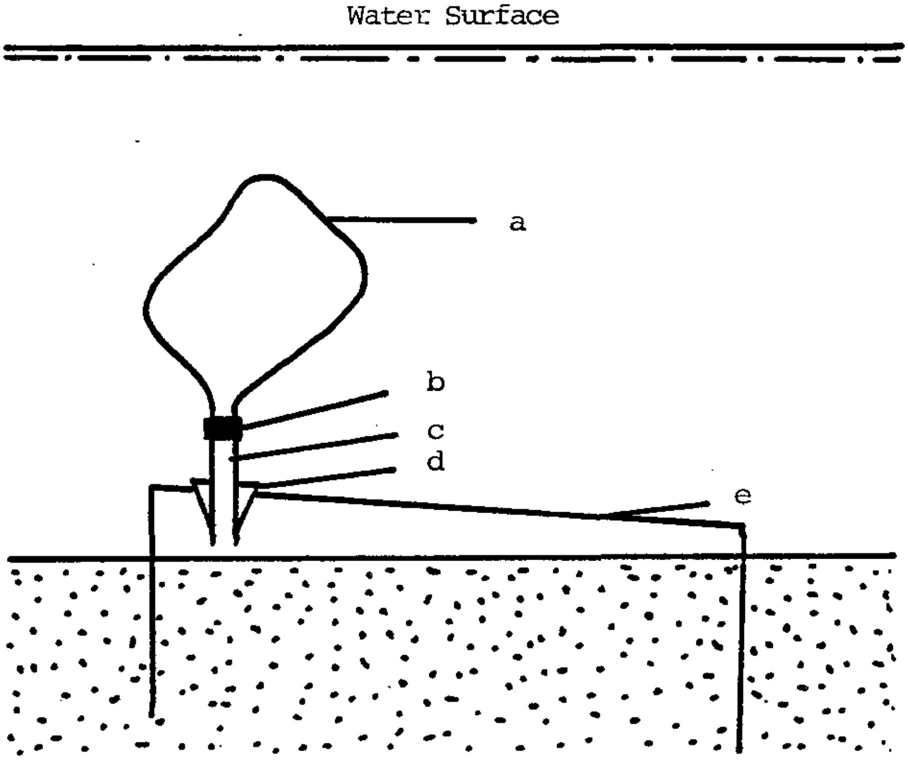


Figure 7. Section view of seepage meter. a - Ziplock heavy duty freezer bag; b - rubber band; c - 1/4" ID polyethylene tube; d - No. 10 rubber stopper; e - end section of steel drum.

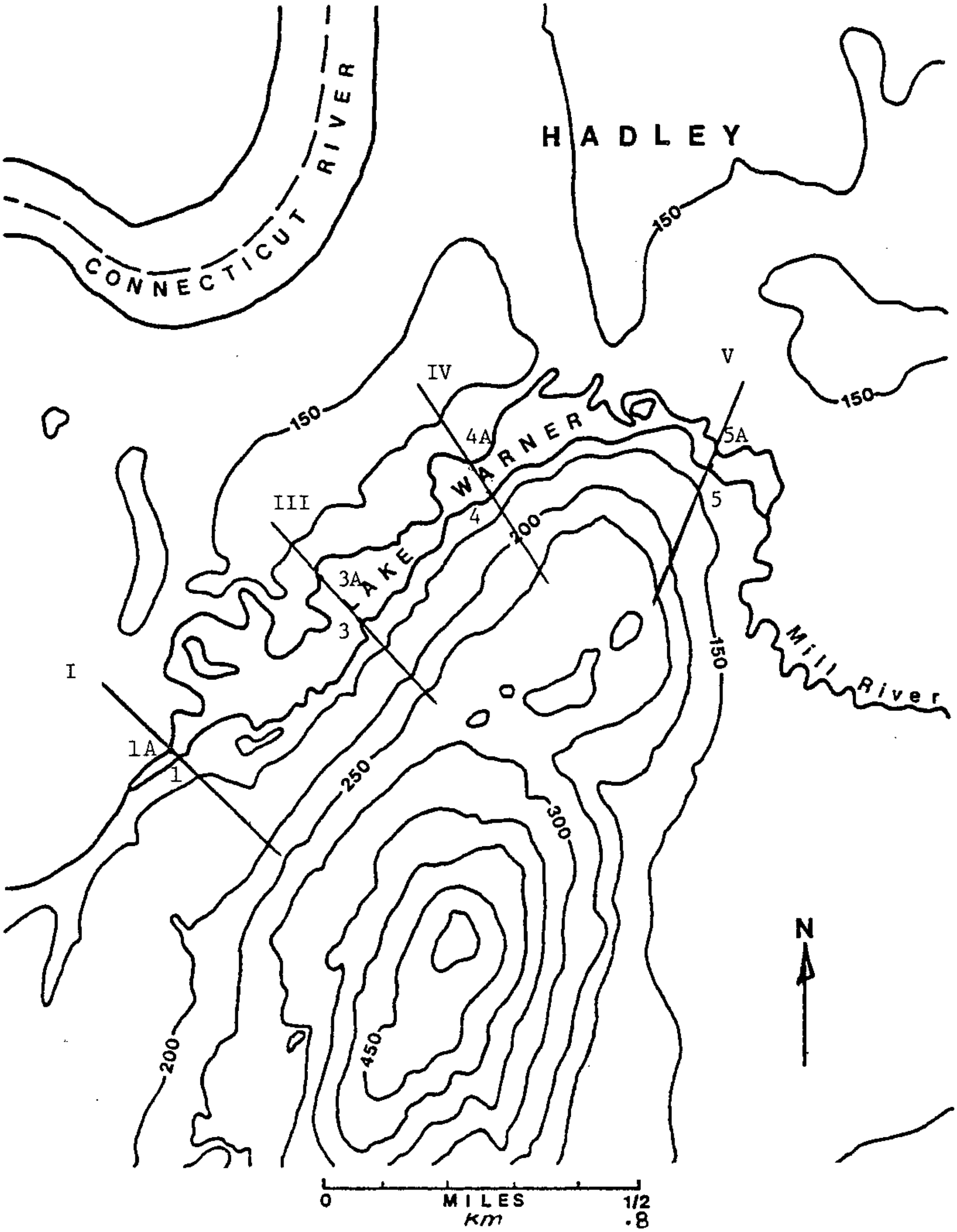


Figure 8. Location map showing seepage meter station and the locations of cross sections.

To estimate seepage rates the net changes in water volume in the sample bags were measured. Measurements were done by weight or volumetrically. Thus if, during the sampling period, there was a net loss of water from the sample bag (each bag was installed with a content of 500 ml of distilled water) then the ground water aquifer was being recharged through the lake sediments. If after the sampling period the sample bag contained greater than the original 500 ml of water then the ground water aquifer was discharging to the lake. These measurements represent a macro-seepage velocity and should not be confused with average interstitial velocities which can be obtained by dividing the seepage rate by the porosity of the sediment. The seepage rates, herein, were estimated by measuring the volume of seepage water (described above) and the duration of the sampling period. Given the area of sediment that was enclosed by the seepage meter the seepage rate can be calculated:

$$u = V/tA \qquad 3.1$$

With volume (V) in liters and time (t) in hours and Area (A) of 0.255 m^2 the seepage rate (u) in μm/sec can be expressed as:

$$u\ (\mu m/sec) = 1.089\ (V/t) \qquad 3.2$$

Discussion

Low flow conditions existed throughout the sampling period in this study. Lake water phosphorus concentration data from July 31, 1981 (Fig. 9) shows a typical dry weather phosphorus profile for surface water and bottom water along an east-west transect through Lake Warner. Data from the intensive sampling period of September 18, 1981, to October 4, 1981 are averaged (Fig. 10) and show a phosphorus profile similar to the wet weather data of Bingham and Feng (1980).

Figure 11 shows the Lake Warner phosphorus budget as determined by stream flow measurements and phosphorus analyses. A net export of phosphorus occurred on the July 31, 1981 sampling date and a net retention of phosphorus occurred during the intensive sampling period (September 18, 1981 to October 4, 1981). Lake sediments released phosphorus during periods of high water temperature and low flow conditions while sediment retention of phosphorus occurs at lower water temperatures and reduced biological activity (Bingham and Feng, 1980). Hence, for Lake Warner, sediment release of phosphorus is seasonal and of short duration, thus only temporarily affecting the nutrient balance of the lake.

The hydrogeologic setting of Lake Warner was examined by constructing cross sections through the lake using a Mount Toby U.S.G.S. quadrangle map and a contour map of the bedrock surface (Londquist, 1974). These cross sections of the bedrock and

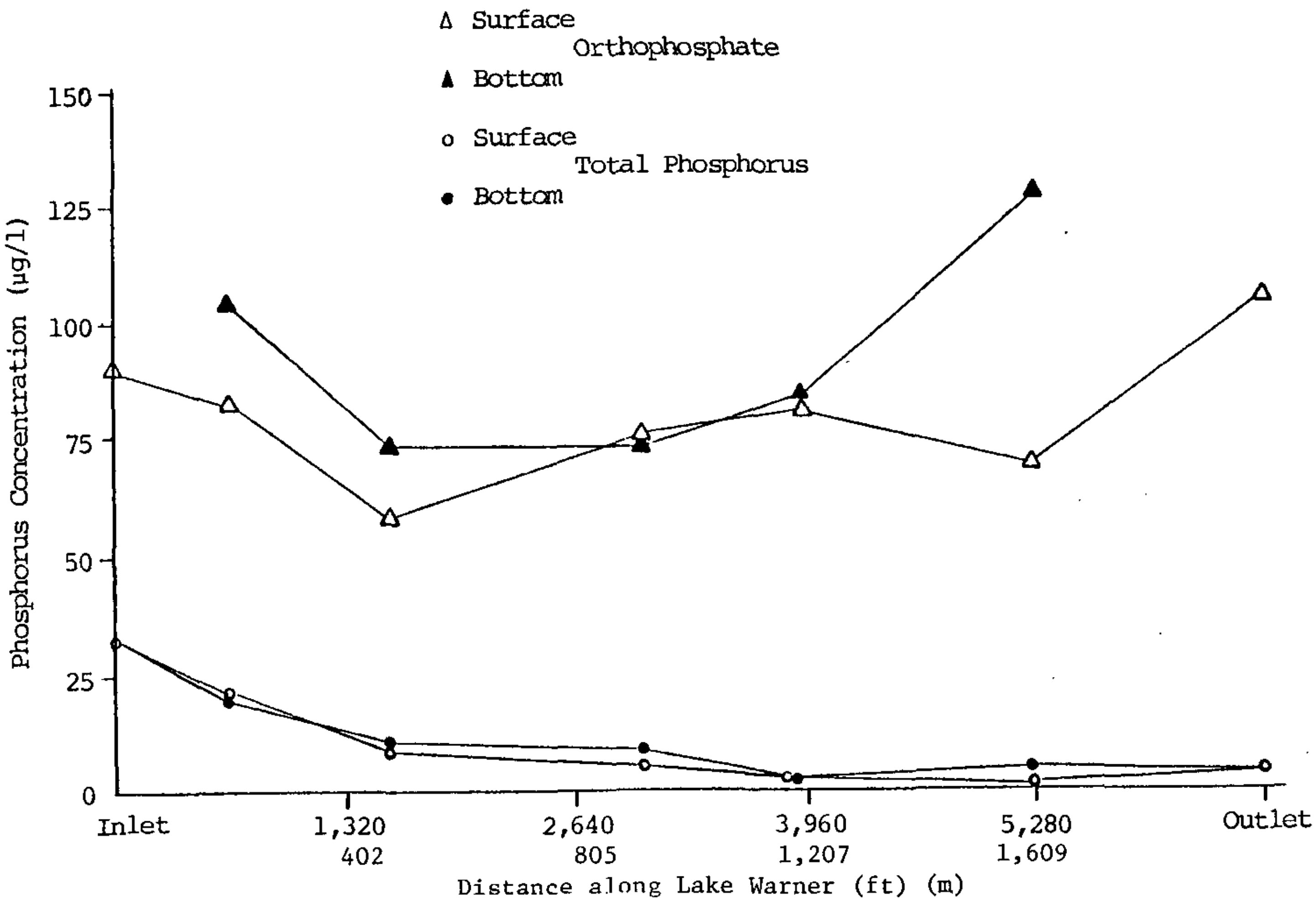


Figure 9. Lake-water phosphorus concentrations for sampling date 7/31/81.

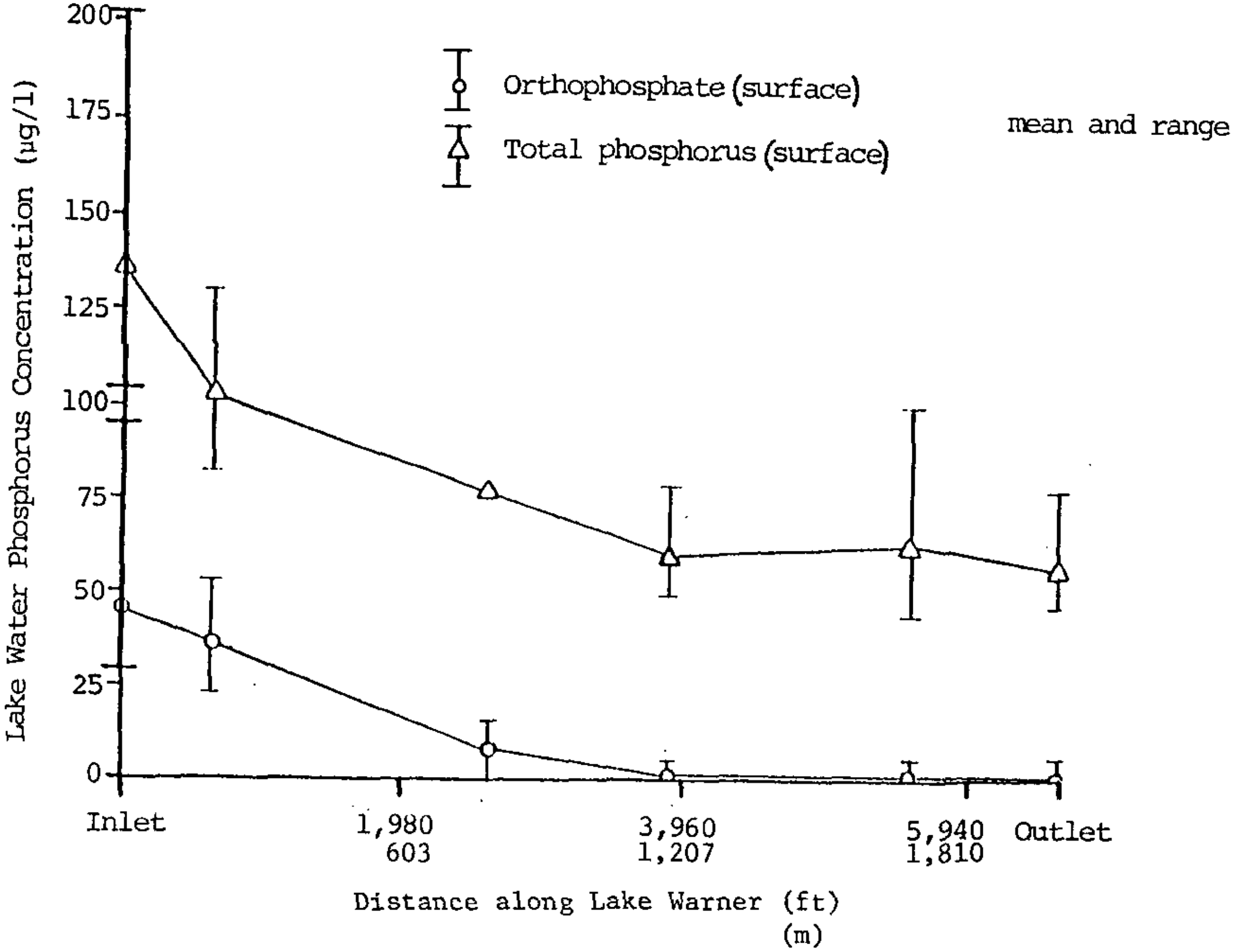


Figure 10. Average lake water phosphorus concentrations for sampling period 9/18/81-10/4/81.

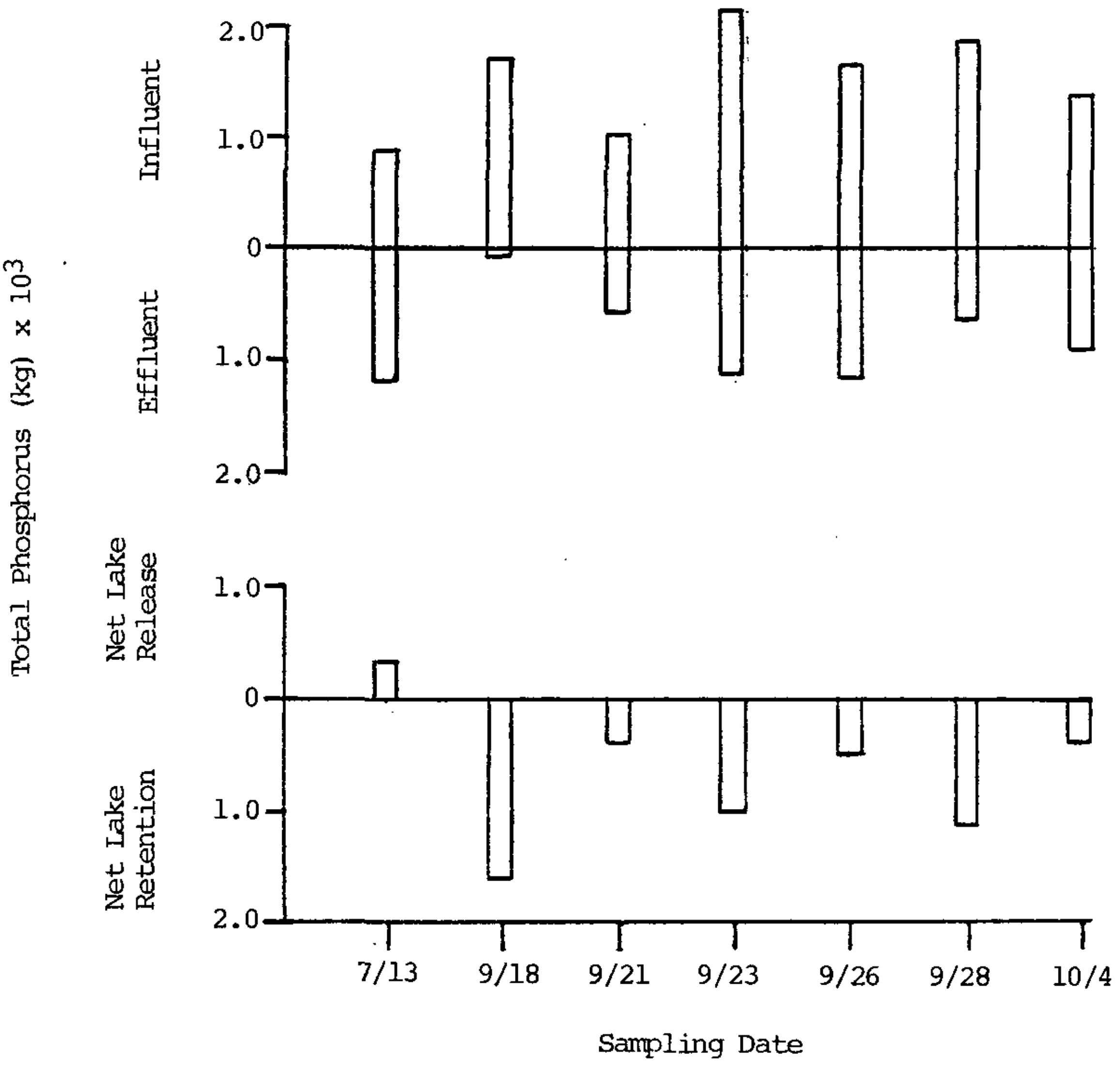


Figure 11. Mass balance on lake water total phosphorus.

overburden elevations appear in Figure 12 and suggest that ground water may discharge into Lake Warner from the southeast and be recharged on the northeast side of the lake.

Figure 13 shows that ground water enters the lake (hence, positive velocities) at meter locations 1, 2, 3, 5 and 5A, and that lake water flows out of the lake at seepage meter locations 1A, 2A, 3A, 4 and 4A. Hence, a flow through condition exists between cross sections I and III. A flow out of the lake occurs at cross section IV and a flow into the lake occurs at cross section V.

Seepage velocity measurements taken at Lake Warner in November 1981 were generally double those taken during July and August 1981. Reversals in seepage flow that occurred at meter locations 1A, 2A, 3A, 4 and 4A were probably due to ground water mounding caused by the precipitation that occurred in October and November 1981 and the reduction of evapotranspiration by vegetation near the lake.

The importance of ground water to Lake Warner's water budget can be examined by using a seepage rate estimated by field measurements ($u_{max} \sim 10^{-2} \mu m/sec$) and assuming that this seepage rate occurs throughout the lake bottom (Area(A) = $2.572(10)^5 m^2$). Then one estimates: Maximum Seepage flow rate = Au $\sim$ 222 m^3/day

Average Mill River flow rate $\sim$ 48902 m^3/day

Comparing the seepage flow rate (222 m^3/day) with the Mill River flow rate (48902 m^3/day) makes it clear that seepage flow can be neglected in the water budget of Lake Warner.

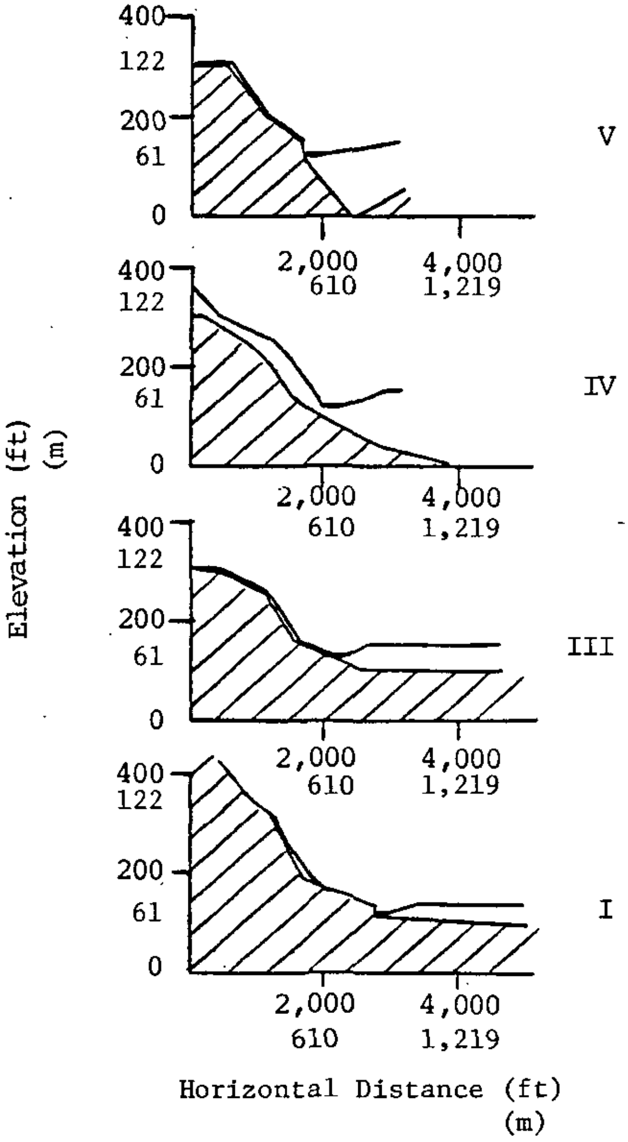


Figure 12. Cross sections through Lake Warner showing the bedrock overburden elevations. The locations of cross sections are shown in Figure 8.

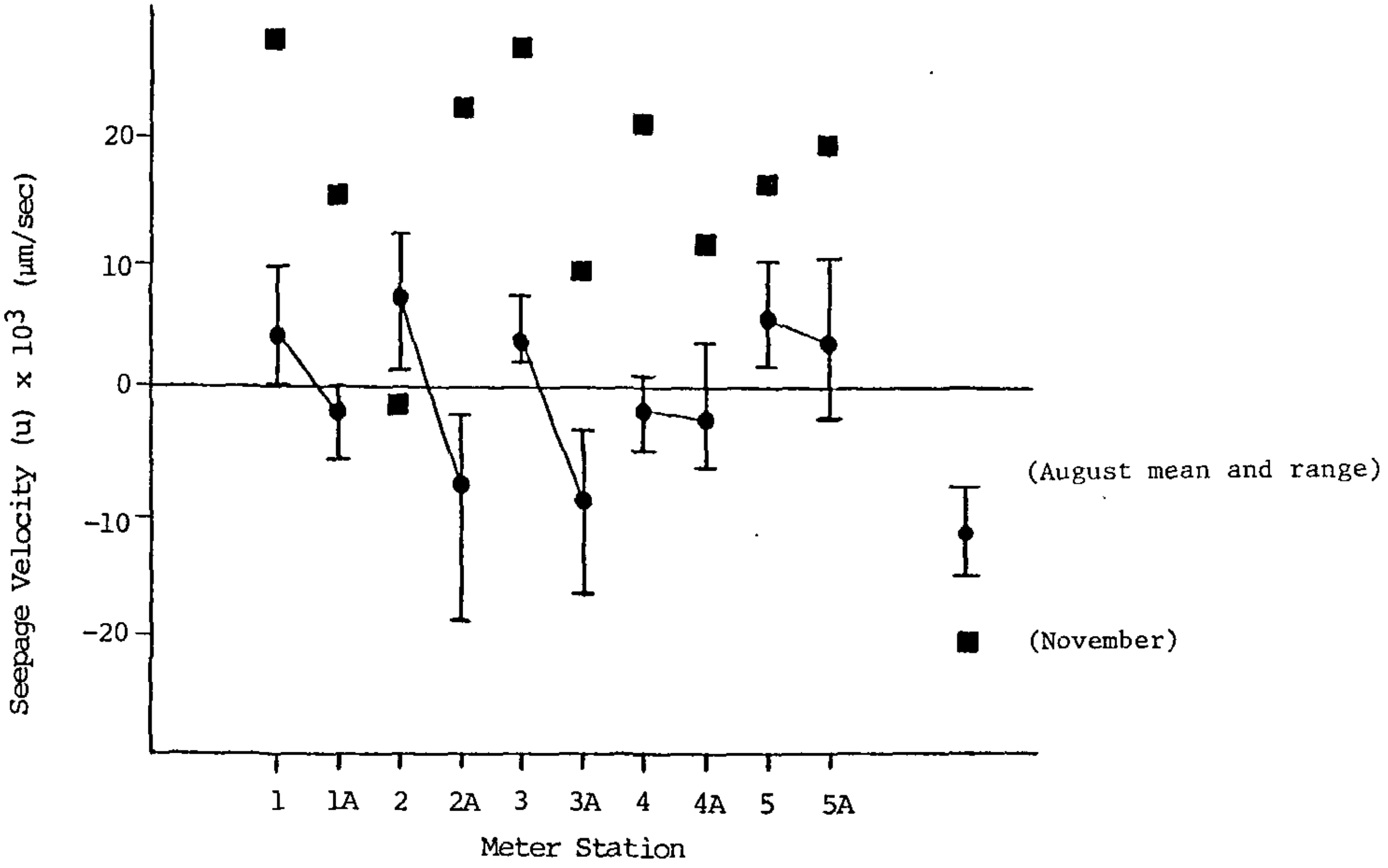


Figure 13. Seepage velocities. Negative seepage velocities mean seepage out of the lake. The station locations are shown in Figure 8.

The significance of ground water discharge through lake sediments as a nutrient transport mechanism can be examined by comparing seepage flux with the phosphorus loading from the Mill River. The average flow rate, determined by Snow and DiGiano (1976), (48902.4 m^3/day) and the average phosphorus concentration of 50 μg/l (as determined by Bingham and Feng, 1980) were used to estimate Mill River phosphorus loading. Average sediment interstitial phosphorus concentrations of 176 μg/l as measured by Bingham and Feng (1980) were used. The maximum value for seepage rates, determined in this study to be about 10^{-2} μm/sec through Lake Warner sediments were assumed to occur uniformly through the lake bottom. This is also a high estimate as seepage was found to recharge the ground water aquifer over part of the lake bottom during this study. One can then estimate the relative phosphorus loadings from the Mill River and from seepage as:

Average Mill River Loading $\sim 2.4(10)^6$ mgP/day

Maximum Seepage Loading $\sim 3.9(10)^4$ mgP/day

As can be seen from the above calculation, the maximum expected seepage loading of phosphorus to Lake Warner is very small ($3.9(10)^4$ mgP/day) compared with the average Mill River phosphorus loading ($2.4(10)^6$ mgP/day). Hence, phosphorus loading due to ground water discharge to the lake can be neglected in the construction of a phosphorus budget for Lake Warner.

CHAPTER IV

LAKE RECOVERY MODEL

The lake recovery model developed by Snow and DiGiano (1976) consists of three control volumes (on lake water phosphorus, and on lake sediment interstitial and sediment solid phase phosphorus) which are shown in Figure 14. Control volume I shows the mechanisms by which phosphorus is transported through the lake water. Snow and DiGiano (1976) showed that Lake Warner more closely approximated a Continuous Flow Stirred Tank Reactor (CFSTR) than a Plug Flow Reactor. Hence, CFSTR assumptions were used in their lake recovery model. These assumptions are:

i) The lake is completely mixed.

ii) The outflow phosphorus concentration is the lake water phosphorus concentration.

iii) The Mill River flow rate (Q) is assumed to be constant on an annual average time scale. The change in lake water storage can therefore be neglected.

iv) The Mill River influent phosphorus concentration P_o is constant.

v) The mass transfer rate (K_1) and the reaction rates (K_2) and (K_3) is constant.

Mass balance equations were developed for the control volumes shown in Figure 14 under the assumption that both total phosphorus and water are conservative substances. The following

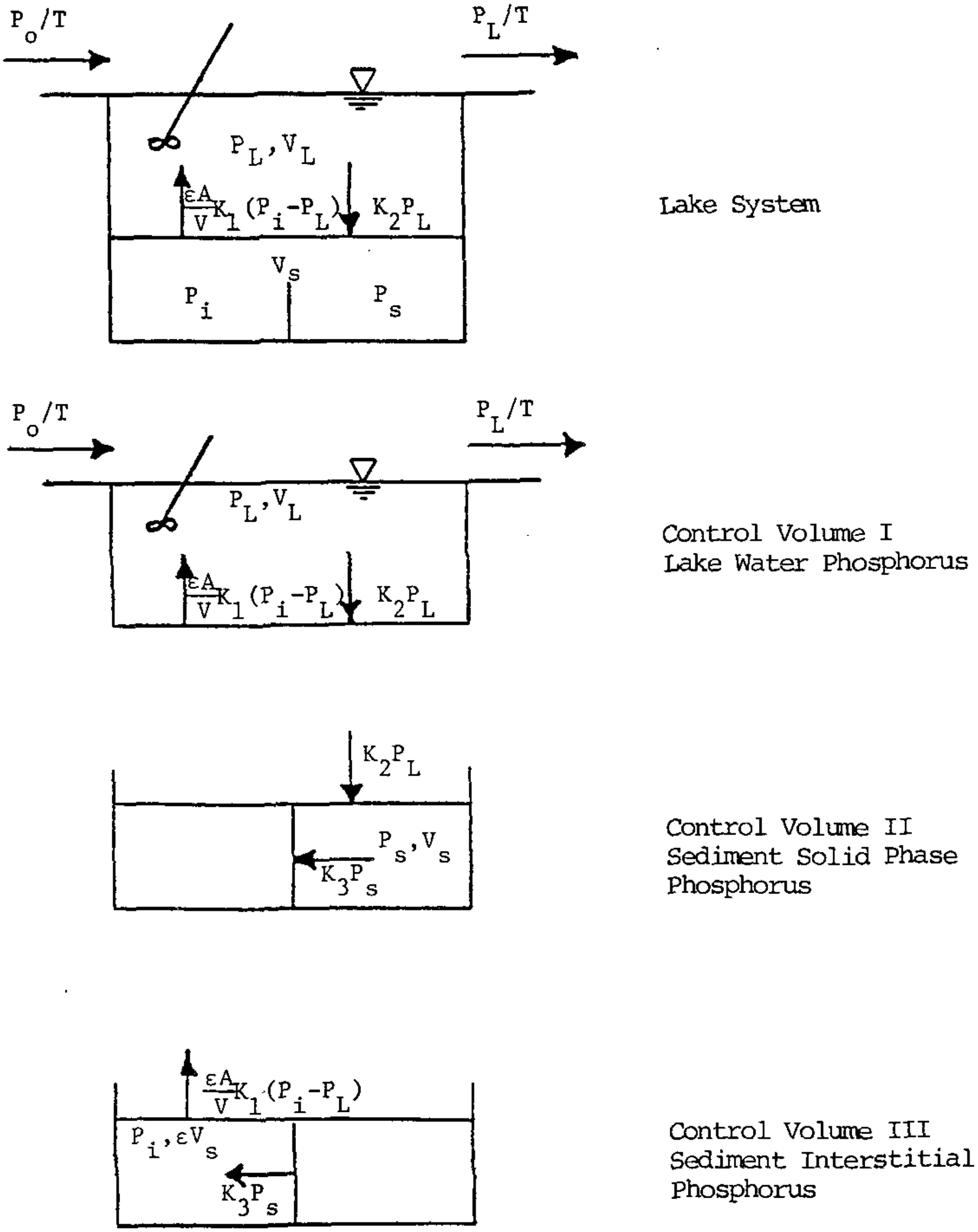


Figure 14. Control volumes used in the lake phosphorus budget model.

discussion will use the term "phosphorus" as synonymous with total phosphorus which is the *conservative substance* being modeled in the lake system.

Release Rate (K_1)

The release of phosphorus from lake bottoms is governed by physical, chemical and biological processes. The rate limiting step for phosphorus release from the sediments could emanate from resistance due to pore diffusion rates through the interstitial water or resistance at the interface of the sediment deposit and the lake water (DiGiano, 1971). Because concentrations of sediment phosphorus in eutrophic lakes is high, it can be assumed that interfacial resistance entirely controls the rate of mass transport. The dominance of the interfacial resistance effect over pore diffusion rates in deeper sediments can be defended by examining the magnitude of the sediment solid phase phosphorus concentration in the upper sediment layers.

The concentration of exchangeable phosphate in Lake Warner sediments is on the order of 10^5 μg/l (Snow and DiGiano, 1976). With an average lake water depth of about 1.7 meters, 0.1 meters of sediment (0.1 meters is the exchange depth used by Snow and DiGiano, 1976) could replace a lake water phosphorus concentration of 100 μg/l about 60 times. Therefore, with an average lake water residence time of about 10 days the sediments alone contain a phosphorus source capable of keeping the lake water at 100 μg/l for 600 days.

DiGiano (1971) suggests a simple model of interfacial resistance to describe the rate of mass transport in deposit-water systems. In the case of lake bottom phosphorus the release is proportional to the difference in the sediment interstitial phosphorus concentration (P_i) and the phosphorus concentration in the overlying water (P_L). The release of phosphorus directly from the solid phase to the lake water is negligible because of the small area of sediment solid phase phosphorus in direct contact with the lake water compared with the area in contact with interstitial water. The exchange depth, estimated by Snow and DiGiano (1976) to be 0.1 meters, multiplied by the lake area is the volume of lake bottom allowed to release phosphorus to the lake water. The volume of sediment solid phase phosphorus (P_s) is represented by V_s (rather than $(1 - \varepsilon)V_s$) and the sediment interstitial phosphorus (P_i) as εV_s. This is appropriate because P_s is three orders of magnitude larger than P_i. Snow and DiGiano (1976) carried out an <u>in situ</u> caisson study at Lake Warner to determine the value of the mass transfer coefficient (K_1) which has a magnitude of about 0.1 meters/day. Hence, the lake bottom phosphorus release can be described as:

$$R_{rel} = \frac{\varepsilon A}{V_s} K_1 (P_i - P_L) \qquad 4.1$$

Sedimentation Rate (K_2)

The removal of phosphorus from the lake water to the sediment can be modeled by assuming a first order reaction rate (K_2) which describes all physical, chemical and biological processes in the lake (Snow and DiGiano, 1976). K_2 has a magnitude of about 0.1/day. The rate of removal of phosphorus from the lake water (assumed to be proportional to the lake water phosphorus concentration) to the sediment can be described as:

$$R_{sed} = K_2 P_L \tag{4.2}$$

Conversion Rate (K_3)

The modification of the Snow and DiGiano (1976) model by Bingham and Feng (1980) to include the reaction rate K_3 required the introduction of a mass balance equation on sediment interstitial phosphorus.

The reaction rate (K_3) describes the conversion of sediment *solid phase phosphorus* (P_s) to sediment interstitial phosphorus (P_i). It is a composite parameter which describes all physical, chemical and biological processes in the sediment. K_3 is assumed to be constant although seasonal fluctuations in K_3 values might be expected due to variations in Mill River water quality and flow rate which affects lake water temperature, dissolved oxygen, biological activity and sediment redox potential.

The rate of conversion from sediment solid phase to interstitial phosphorus is assumed to be proportional to the sediment solid phase phosphorus concentration.

$$\begin{array}{l} \text{Rate of conversion from solid} \\ \text{to interstitial phosphorus in} \\ \text{sediment.} \end{array} = K_3 P_s \qquad 4.3$$

Mass Balance Equations

The basic equation for the conservation of a reactive material is:

$$\begin{array}{c} \text{Rate of} \\ \text{Mass Input} \end{array} - \begin{array}{c} \text{Rate of} \\ \text{Mass Output} \end{array} \pm \begin{array}{c} \text{Rates of} \\ \text{Reaction} \end{array} = \begin{array}{c} \text{Rate of} \\ \text{Accumulation} \end{array} \qquad 4.4$$

The terms of the above mass balance equation for the lake water phosphorus control volume (control volume I in Fig. 14) become:

$$\text{Mass Input Rate} = QP_o + \epsilon A K_1 (P_i - P_L) \qquad 4.5$$

$$\text{Mass Output Rate} = QP_L \qquad 4.6$$

$$\text{Rate of Reaction} = V_L K_2 P_L \qquad 4.7$$

$$\begin{array}{l} \text{Rate of Mass} \\ \text{Accumulation} \end{array} = V_L \frac{\Delta P_L}{\Delta t} \qquad 4.8$$

Substituting equations (4.5,6,7 and 8) into equation (4.4) and letting the deltas go to zero yields:

$$V_L \frac{dP_L}{dt} = QP_0 - QP_L - V_L K_2 P_L + \varepsilon A K_1 (P_i - P_L) \qquad 4.9$$

The terms for the mass balance equation for the sediment solid phase phosphorus control volume (control volume II in Fig. 14) become:

Mass input rate $= V_L K_2 P_L$ 4.10

Mass output rate $= V_s K_3 P_s$ 4.11

Rate of Mass Accumulation $= V_s \frac{\Delta P_s}{\Delta t}$ 4.12

Substituting equations (4.10, 11 and 12) into equation (4.4) and letting the deltas go to zero yields:

$$V_s \frac{dP_s}{dt} = V_L K_2 P_L - V_s K_3 P_s \qquad 4.13$$

The terms for the mass balance equation for the sediment interstitial phosphorus control volume (control volume III in Fig. 14) become:

Mass Input Rate $= V_s K_3 P_s$ 4.14

Mass Output Rate $= \varepsilon A K_1 (P_i - P_L)$ 4.15

Rate of Mass Accumulation $= \varepsilon V_s \frac{\Delta P_i}{\Delta t}$ 4.16

Substituting equations (4.14, 15 and 16) into equation (4.4) and letting the deltas go to zero yields:

$$V_s \frac{dP_i}{dt} = V_s K_3 P_s - \varepsilon A K_1 (P_i - P_L) \qquad 4.17$$

Hence, the modified Snow and Digiano (1976) model as presented by Bingham and Feng (1980) is:

Sediment Solid Phase Phosphorus

$$V_s \frac{dP_s}{dt} = -V_s K_3 P_s + V_L K_2 P_L \qquad 4.13$$

Sediment Interstitial Phosphorus

$$V_s \frac{dP_i}{dt} = V_s K_3 P_s - \varepsilon A K_1 (P_i - P_L) \qquad 4.17$$

Lake Water Phosphorus

$$V_L \frac{dP_L}{dt} = QP_o - QP_L + \varepsilon A K_1 (P_i - P_L) - V_L K_2 P_L \qquad 4.9$$

The above equations can be rewritten in a form which is more convenient for solution by grouping the coefficients:

Sediment Solid Phase Phosphorus

$$\frac{dP_s}{dt} = X_1 P_s + X_2 P_L \qquad 4.18$$

Sediment Interstitial Phosphorus

$$\frac{dP_i}{dt} = X_3 P_s + X_4 P_i + X_5 P_L \qquad 4.19$$

Lake Water Phosphorus

$$\frac{dP_L}{dt} = X_6P_i + X_7P_L + X_8 \qquad 4.20$$

where:

$X_1 = -K_3$

$X_2 = K_2V_L/V_s$

$X_3 = K_3/\varepsilon$

$X_4 = -AK_1/V_s$

$X_5 = AK_1/V_S$

$X_6 = \varepsilon AK_1/V_L$

$X_7 = -(Q + \varepsilon AK_1 + K_2V_L)/V_L$

$X_8 = P_oQ/V_L$

This system of ordinary, first order, linear differential equations with constant coefficients (equations 4.18, 19, and 20) can be solved numerically (Bingham and Feng, 1980) or analytically. Analytical methods of solving these equations were explored including the Laplace transform method, the power series method, the trial function method and approximation methods. The Laplace transform method was not chosen because the inverse transform appeared formidable. The power series method was rejected after calculations using a digital computer showed the series did not approach convergence after evaluating one hundred terms.

The trial function method was successful in leading to the solution of equations (4.18, 19 and 20). Application of approximation techniques allowed these equations to be simplified sufficiently so that they could be solved readily. Both the trial function method and the approximation techniques are discussed more fully in the following sections.

Trial Function Method

The trial function method (Stockton, personal communication) is useful for solving a small number of simultaneous ordinary first order linear differential equations with constant coefficients. The number of algebraic equations to be manipulated to determine unknown coefficients grows exponentially with the number of simultaneous differential equations to be solved so that three simultaneous differential equations may be a practical upper limit for the method. The lake model requires the simultaneous solution of 12 algebraic equations. The non-homogeneous differential equations in the lake model were converted to homogeneous equations to simplify algebraic manipulations. Details of the trial function method appear in Appendix B. The equations for the lake water ($P_L(t)$), sediment interstitial ($P_i(t)$) and sediment solid phase ($P_s(t)$) phosphorus concentration as determined by the trial function method are:

$$
\begin{aligned}
P_L(t) = {} & P_{L\infty} + (P_{Lo} - P_{L\infty}) \Big[\Big[E(BF-CE)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)] \Big] e^{\alpha t} + \\
& \Big[F(CD-AF)/[(F-E)(CD-AF) - (F-D)(CE-BF)] \Big] e^{\beta t} + \\
& \Big[D(AE-DB)/[(D-F)(AE-BD) - (D-E)(AF-CD)] \Big] e^{\gamma t} \Big] \\
& + (P_{io} - P_{i\infty}) \Big[\Big[E(E-F)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)] \Big] e^{\alpha t} + \\
& \Big[F(F-D)/[(F-E)(CD-AF) - (F-D)(CE-BF)] \Big] e^{\beta t} + \\
& \Big[D(D-E)/[(D-F)(AE-BD) - (D-E)(AF-CD)] \Big] e^{\gamma t} \Big] \\
& - (P_{so} - P_{s\infty}) \Big[\Big[E(B-C)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)] \Big] e^{\alpha t} + \\
& \Big[F(C-A)/[(F-E)(CD-AF) - (F-D)(CE-BF)] \Big] e^{\beta t} + \\
& \Big[D(A-B)/[(D-F)(AE-BD) - (D-E)(AF-CD)] \Big] e^{\gamma t} \Big] .
\end{aligned}
\tag{4.21}
$$

To use equation (4.21) substitute the values for the initial and steady state conditions and make the following substitutions:

$$A = [(\alpha - X_4)(\alpha - X_7) - X_5X_6]/X_3X_6 \tag{4.21.1}$$

$$B = [(\beta - X_4)(\beta - X_7) - X_5X_6]/X_3X_6 \tag{4.21.2}$$

$$C = [(\gamma - X_4)(\gamma - X_7) - X_5X_6]/X_3X_6 \tag{4.21.3}$$

$$D = [X_2X_3 + X_5\ (\alpha - X_4)]/(\alpha - X_4)(\alpha - X_1) \tag{4.21.4}$$

$$E = [X_2X_3 + X_5\ (\beta - X_4)]/(\beta - X_4)(\beta - X_1) \tag{4.21.5}$$

$$F = [X_2X_3 + X_5\ (\gamma - X_4)]/(\gamma - X_4)(\gamma - X_1) . \tag{4.21.6}$$

$$
\begin{aligned}
P_i(t) = P_{i\infty} + {} & (P_{io} - P_{i\infty}) \Bigg[\bigg[E(BF-CE)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)] \bigg] e^{\alpha t} + \\
& \Big[F(CD-AF)/[(F-E)(CD-AF) - (F-D)(CE-BF)] \Big] e^{\beta t} + \\
& \Big[D(AE-DB)/[(D-F)(AE-BD) - (D-E)(AF-CD)] \Big] e^{\gamma t} \Bigg] \\
& + (P_{so} - P_{s\infty}) \Bigg[\bigg[E(E-F)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)] \bigg] e^{\alpha t} + \\
& \Big[F(F-D)/[(F-E)(CD-AF) - (F-D)(CE-BF)] \Big] e^{\beta t} + \\
& \Big[D(D-E)/[(D-F)(AE-BD) - (D-E)(AF-CD)] \Big] e^{\gamma t} \Bigg] \\
& - (P_{Lo} - P_{L\infty}) \Bigg[\bigg[E(B-C)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)] \bigg] e^{\alpha t} + \\
& \Big[F(C-A)/[(F-E)(CD-AF) - (F-D)(CE-BF)] \Big] e^{\beta t} + \\
& \Big[D(A-B)/[(D-F)(AE-BD) - (D-E)(AF-CD)] \Big] e^{\gamma t} \Bigg] \qquad 4.22
\end{aligned}
$$

To use equation (4.22) substitute the values for the initial and steady state conditions and make the following substitutions:

$$A = X_2X_6/(\alpha - X_1)(\alpha - X_7) \qquad 4.22.1$$

$$B = X_2X_6/(\beta - X_1)(\beta - X_7) \qquad 4.22.2$$

$$C = X_2X_6/(\gamma - X_1)(\gamma - X_7) \qquad 4.22.3$$

$$D = [(\alpha - X_1)(\alpha - X_4)(\alpha - X_7) - X_2X_3X_6]/X_5(\alpha - X_1)(\alpha - X_7) \qquad 4.22.4$$

$$E = [(\beta - X_1)(\beta - X_4)(\beta - X_7) - X_2X_3X_6]/X_5(\beta - X_1)(\beta - X_7) \qquad 4.22.5$$

$$F = [(\gamma - X_1)(\gamma - X_4)(\gamma - X_7) - X_2X_3X_6]/X_5(\gamma - X_1)(\gamma - X_7) \qquad 4.22.6$$

$$\begin{aligned}
P_s(t) = P_{s\infty} + (P_{so} - P_{s\infty}) \Big[& \big[E(BF-CE)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)]\big] e^{\alpha t} + \\
& \big[F(CD-AF)/[(F-E)(CD-AF) - (F-D)(CE-BF)]\big] e^{\beta t} + \\
& \big[D(AE-DB)/[(D-F)(AE-BD) - (D-E)(AF-CD)]\big] e^{\gamma t} \Big] \\
+ (P_{Lo} - P_{L\infty}) \Big[& \big[E(E-F)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)]\big] e^{\alpha t} + \\
& \big[F(F-D)/[(F-E)(CD-AF) - (F-D)(CE-BF)]\big] e^{\beta t} + \\
& \big[D(D-E)/[(D-F)(AE-BD) - (D-E)(AF-CD)]\big] e^{\gamma t} \Big] \\
- (P_{io} - P_{i\infty}) \Big[& \big[E(B-C)/ \\
& [(E-D)(BF-CE) - (E-F)(BD-AE)]\big] e^{\alpha t} + \\
& \big[F(C-A)/[(F-E)(CD-AF) - (F-D)(CE-BF)]\big] e^{\beta t} + \\
& \big[D(A-B)/[(D-F)(AE-BD) - (D-E)(AF-CD)]\big] e^{\gamma t} \Big]
\end{aligned} \tag{4.23}$$

To use equation (4.23) substitute the values for the initial and steady state conditions and make the following substitutions:

$$A = (\alpha - X_1) \tag{4.23.1}$$

$$B = (\beta - X_1) \tag{4.23.2}$$

$$C = (\gamma - X_1) \tag{4.23.3}$$

$$D = X_3/[\alpha - X_4 - X_6X_7/(\alpha - X_7)] \tag{4.23.4}$$

$$E = X_3/[\beta - X_4 - X_6X_7/(\beta - X_7)] \tag{4.23.5}$$

$$F = X_3/[\gamma - X_4 - X_6X_7/(\gamma - X_7)] \tag{4.23.6}$$

In the above equations α, β and γ are the roots (r_1, r_2 and r_3) of the following equation.

$$r^3 - (X_1 + X_4 + X_7)r^2 + (X_1X_4 + X_1X_7 + X_4X_7 - X_4X_6)r + X_1X_4X_6 - X_1X_4X_7 - X_2X_3X_6 = 0 \qquad 4.24$$

and

$$P_{L\infty} = P_o$$

$$P_{i\infty} = P_oX_2/X_5\varepsilon + P_o$$

$$P_{s\infty} = -P_oX_2/X_1$$

These equations appear formidable but can be programmed on a TI-59 programmable calculator. The program coding for the equations (4.21, 22 and 23) appears in Appendix C. It is suggested that terms that contain βt and γt in the exponent position be neglected in applications of the equations. β and γ are very large ($O = [10^{-1}]$ and $O = [1]$, respectively) compared to α ($O = [10^{-3}]$). Hence as t grows large ($t > 100$ days) terms with large negative rate constants (β and γ) in the exponent go to zero quickly and can be neglected.

Approximation Techniques

Short term predictions. Over short time steps the value of the interstitial phosphorus concentration can be considered constant (DiGiano and Snow, 1976). The value of P_i in equation

(4.9) is large relative to the other terms (see order of magnitude analysis in Appendix B) and can therefore be considered constant. This is similar to pseudo-order assumptions which allow species in excess in a chemical reaction to be considered constant to simplify the expression for the differential rate equation. In the initial period ($t < 3T$) after phosphorus loading is reduced, dilution plays an important role in lake recovery. Hence, holding P_i constant and integrating equation (4.9) to give equation (4.25) yields a short term model that is most influenced by the dilution process but allows for phosphorus release from the lake bottom.

$$P_L(t) = -(X_8 + X_6P_{io})/X_7 + [(X_8 + X_6P_{io})/X_7 + P_{Lo}]e^{X_7t} \qquad 4.25$$

Figure 15 compares predictions of the short term model (equation 4.25) with predictions of the trial function solution (equation 4.21). Figure 16 compares the same equations on an expanded time scale. It is evident that the short term predictions are accurate for only two or three hydraulic residence times (the average annual hydraulic residence time for Lake Warner is 10 days).

After the initial rapid reduction of the lake phosphorus concentration due to dilution, phosphorus release from the lake bottom may control lake recovery.

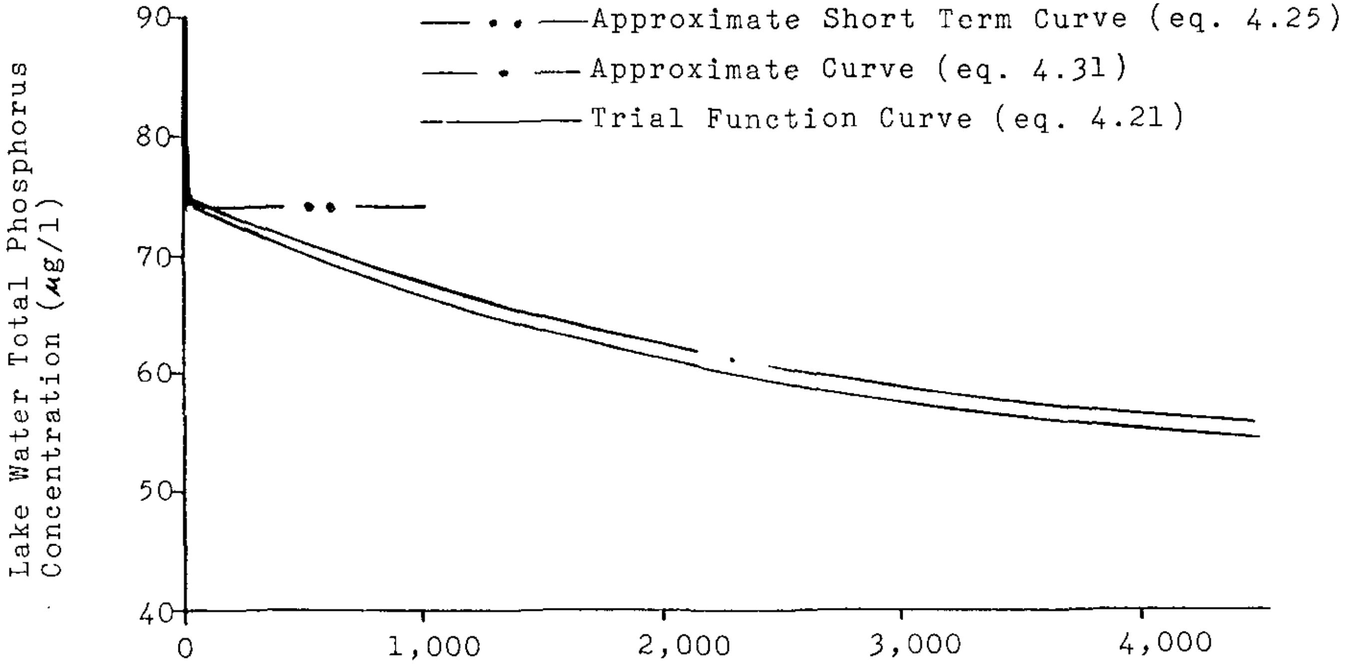


Figure 15. Comparison of model predictions ($P_L(t)$), Trial Function Solution (eq. 4.21), Approximate Short-Term Cruve (eq. 4.25) and Approximate Curve (eq. 4.31).

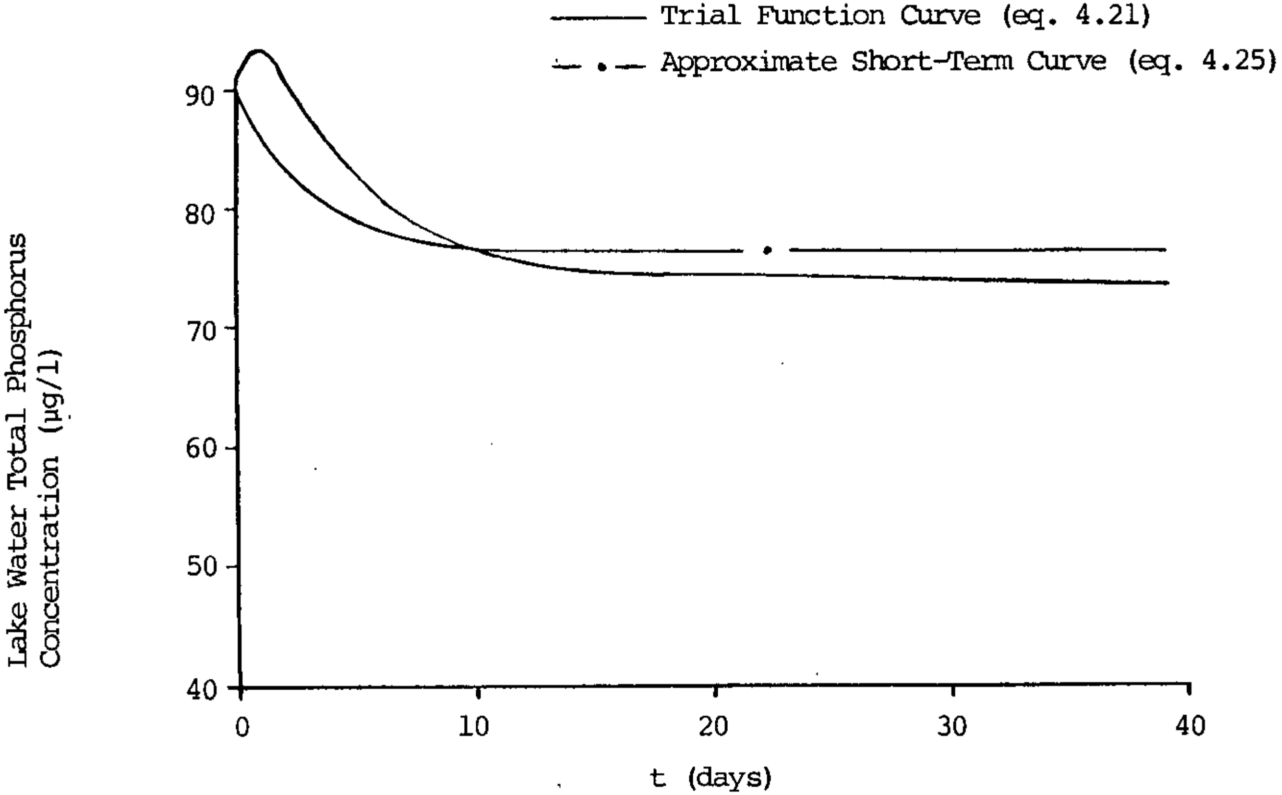


Figure 16. Expanded time scale. Comparison of model preditions ($P_L(t)$), Trial Function Solution (eq. 4.21) and Approximate Solution (eq. 4.25).

Intermediate term predictions. Figure 17, obtained from equation (4.21), shows that after the initial recovery period the lake water phosphorus concentration reaches a quasi-steady state condition. The rate of recovery of the lake is controlled by phosphorus release from the lake bottom, hence, sediment interstitial and solid phase phosphorus concentrations are allowed to change over long time horizons (several years to a decade). In a similar manner in which intermediate chemical species are considered constant to simplify differential rate equations the rate of change of the lake water phosphorus concentration can be considered zero. An algebraic expression for P_L can be found by setting the left hand side of equation (4.20) equal to zero and solving for P_L.

$$P_L = -X_8/X_7 - X_6P_i/X_7 \qquad 4.26$$

Substituting equation (4.26) into equations (4.18) and (4.19) yields:

$$\frac{dP_s}{dt} = X_1P_s - X_2X_8/X_7 - X_2X_6P_i/X_7 \qquad 4.27$$

$$\frac{dP_i}{dt} = X_3P_s + P_i(X_4 - X_5X_6/X_7) - X_5X_8/X_7 \qquad 4.28$$

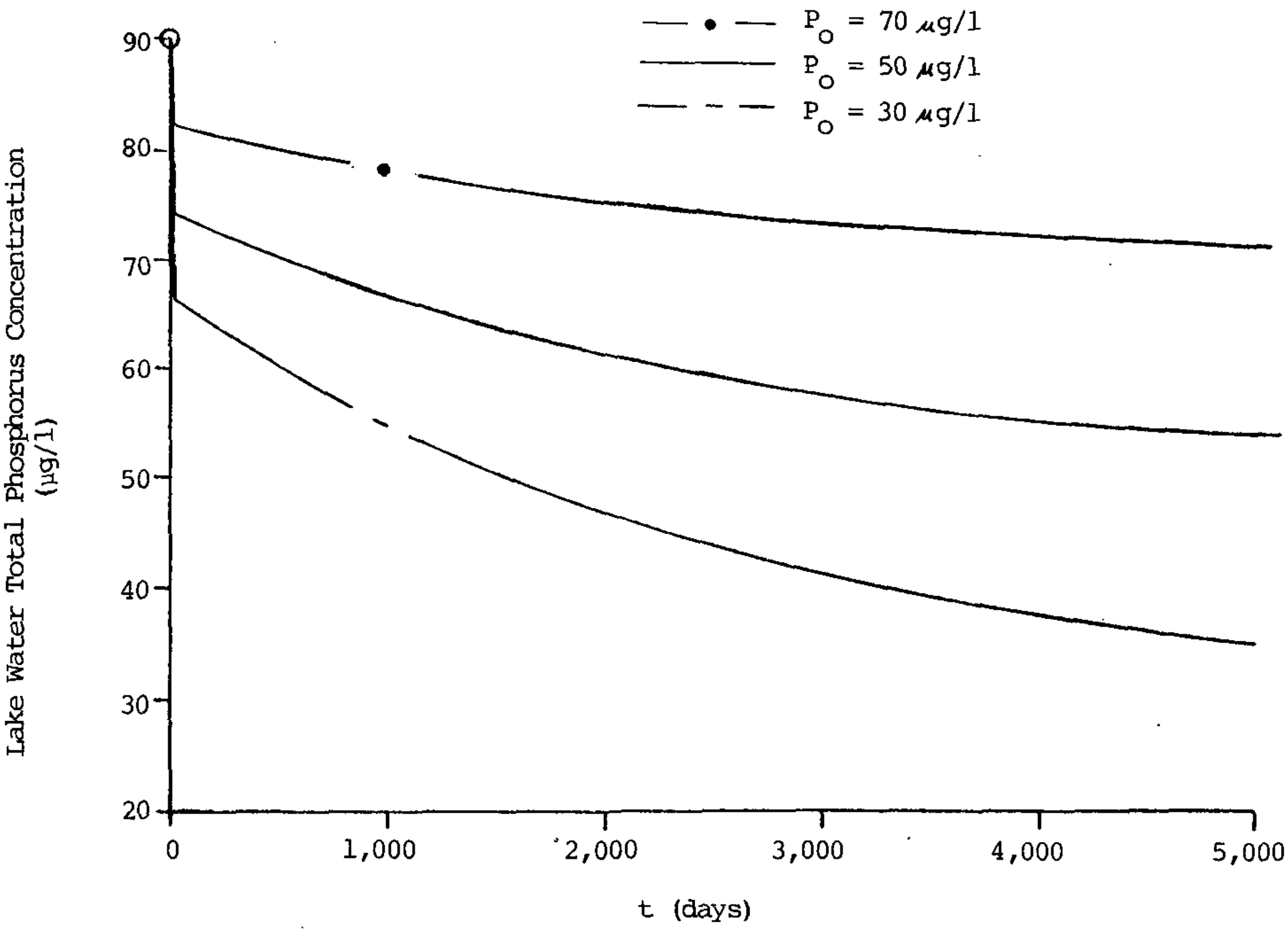


Figure 17. Lake water response to reductions of phosphorus loading (P_o) from 90 to 70, 50 and 30 (μg/l).

Equations (4.27) and (4.28) can be solved simultaneously by Laplace transform methods to yield:

$$\begin{aligned}
P_i(t) = {} & \Big[[P_{so}X_3 - P_{io}(G-r)]/2r\Big]\, e^{-t(H-r)} \\
- {} & \Big[[P_{so}X_3 - P_i(G+r)]/2r\Big]\, e^{-t(H+r)} \\
+ {} & \Big[[X_2X_3X_8/X_7 - X_5X_8/X_7(G+r)]/2r(H+r)\Big][1 - e^{-t(H+r)}] \\
- {} & \Big[[X_2X_3X_8/X_7 - X_5X_8/X_7(G-r)]/2r(H-r)\Big][1 - e^{-t(H-r)}]
\end{aligned}$$

4.29

$$\begin{aligned}
P_s(t) = {} & \Big[[P_{so}(G+r) - P_{io}X_2X_6/X_7]/2r\Big]\, e^{-t(H-r)} \\
- {} & \Big[[P_{so}(G-r) - P_{io}X_2X_6/X_7]/2r\Big]\, e^{-t(H+r)} \\
- {} & \Big[X_2X_8(X_6X_5/X_7^2 - 1)(G-r)/2r(H+r)X_7\Big][1 - e^{-t(H+r)}] \\
+ {} & \Big[X_2X_8(X_6X_5/X_7^2 - 1)(G+r)/2r(H-r)X_7\Big][1 - e^{-t(H-r)}]
\end{aligned}$$

4.30

Substituting the equation for $P_i(t)$ (equation (4.29)) into equation (4.26) yields a time dependent equation for P_L:

$$\begin{aligned}
P_L(t) = {} & -X_8/X_7 - [X_6/X_7]\Big[\Big[[P_{so}X_3 - P_{io}(G-r)]/2r\Big]\, e^{-t(H-r)} \\
- {} & \Big[[P_{so}X_3 - P_i(G+r)]/2r\Big]\, e^{-t(H+r)} \\
+ {} & \Big[[X_2X_3X_8/X_7 - X_5X_8/X_7(G+r)]/2r(H+r)\Big][1 - e^{-t(H+r)}] \\
- {} & \Big[[X_2X_3X_8/X_7 - X_5X_8/X_7(G-r)]/2r(H-r)\Big][1 - e^{-t(H-r)}]\Big]
\end{aligned}$$

4.31

where:

$$G = (X_5X_6 - X_4X_7 + X_1X_7)/2X_7$$

$$H = (X_5X_6 - X_4X_7 - X_1X_7)/2X_7$$

$$r = \Big[[(X_5X_6 - X_4X_7 - X_1X_7)/X_7]^2 - 4\ [X_2X_3X_6/X_7 - X_1\ (X_5X_6 - X_4X_7)/X_7]\Big]^{1/2}/2$$

and

$$X_1 = -K_3$$

$$X_2 = K_2V_L/V_s$$

$$X_3 = K_3/\varepsilon$$

$$X_4 = -AK_1/V_s$$

$$X_5 = AK_1/V_s$$

$$X_6 = \varepsilon AK_1/V_L$$

$$X_7 = -(Q + \varepsilon AK_1 + K_2V_L)/V_L$$

$$X_8 = P_o/T$$

Figures 15, 18 and 19 compare the approximate lake model equations with the solutions obtained from the trial function method. The approximate curves compare favorably with the curves produced by the analytical equations obtained from the trial function method.

Program coding for the use of equations (4.29, 30 and 31) on a TI-59 programmable calculator appear in Appendix C.

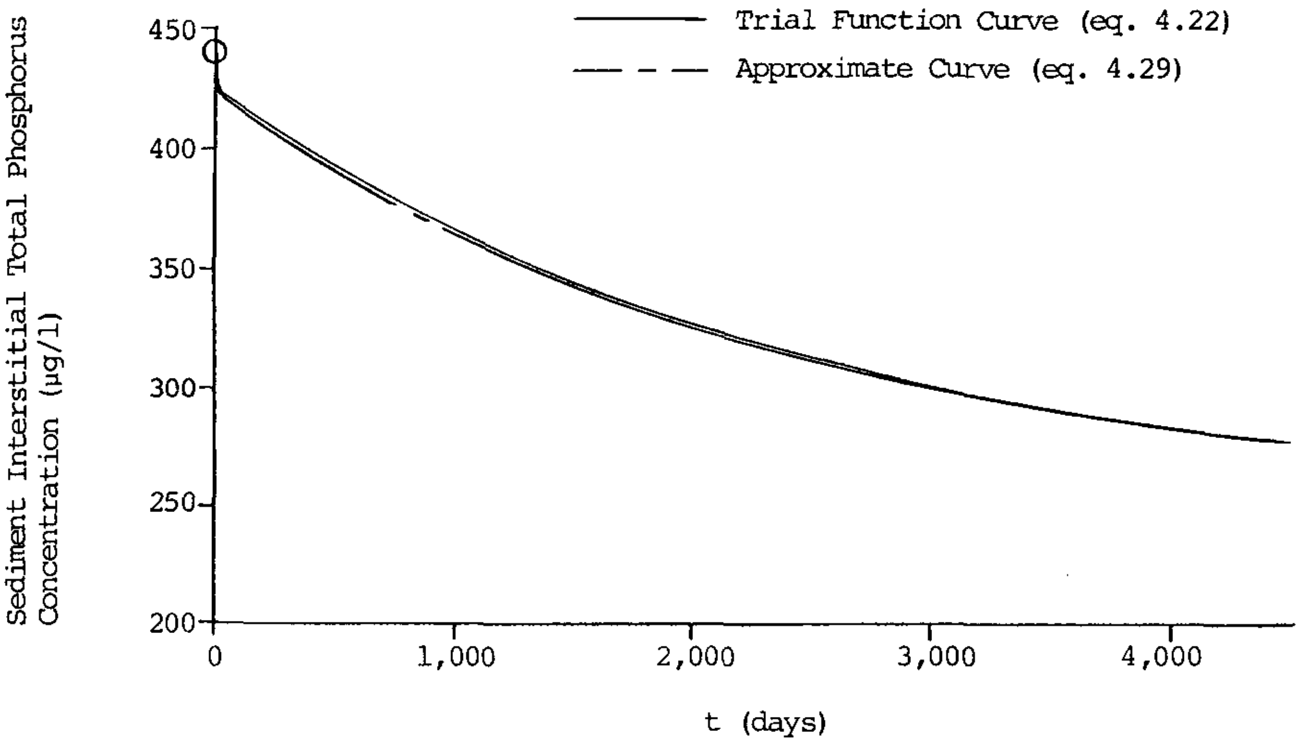


Figure 18. Comparison of model predictions, Trial Function Solution (eq. 4.22) and Approximate Solution (eq. 4.29).

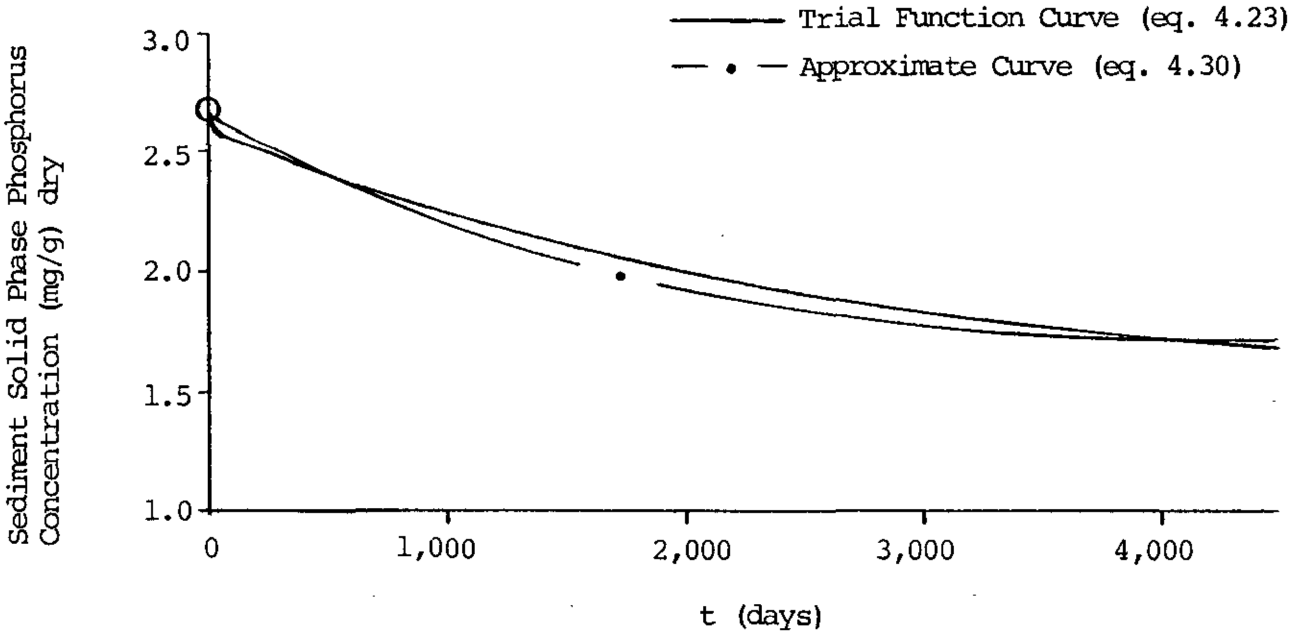


Figure 19. Comparison of model predictions ($P_s(t)$), Trial Function Solution (eq. 4.23) and Approximate Solution (eq. 4.30).

Sensitivity Analysis

Figures 20 through 24 show sensitivity of the lake model equations (4.21, 22 and 23) to parameters K_1, K_2, K_3, Q and Dr. Table 3 shows the initial conditions and parameter values used in the sensitivity analysis. Figure 20 shows that variations of the release rate K_1 by two orders of magnitude has little effect on the prediction of lake water phosphorus concentration. An explanation might be that the sediment desorption rate K_3 controls the release of phosphorus from the lake bottom to the lake water. Sediment solid phase and interstitial phosphorus concentrations are sensitive to variations as small as 1% in the value of K_1.

Figure 21 shows that a 5% variation in the sedimentation rate (K_2) affects the lake water phosphorus concentration in the early stages of the lake recovery. The effect of variations of K_2 diminishes as the time horizon increases. In contrast, varying K_2 affects long term predictions of the sediment solid phase and interstitial phosphorus concentrations.

Figures 22 and 23 show that the sensitivity of the lake water equation to the phosphorus desorption rate (K_3) and sediment reactive depth (Dr) is comparable to the sensitivity to K_2. The value of K_2, K_3, or Dr used is most important when making predictions of less than five years. The sediment interstitial phosphorus concentration is significantly effected by varying the values of K_3 and Dr ($t < 5$ years) but grows sensitive to K_2 for

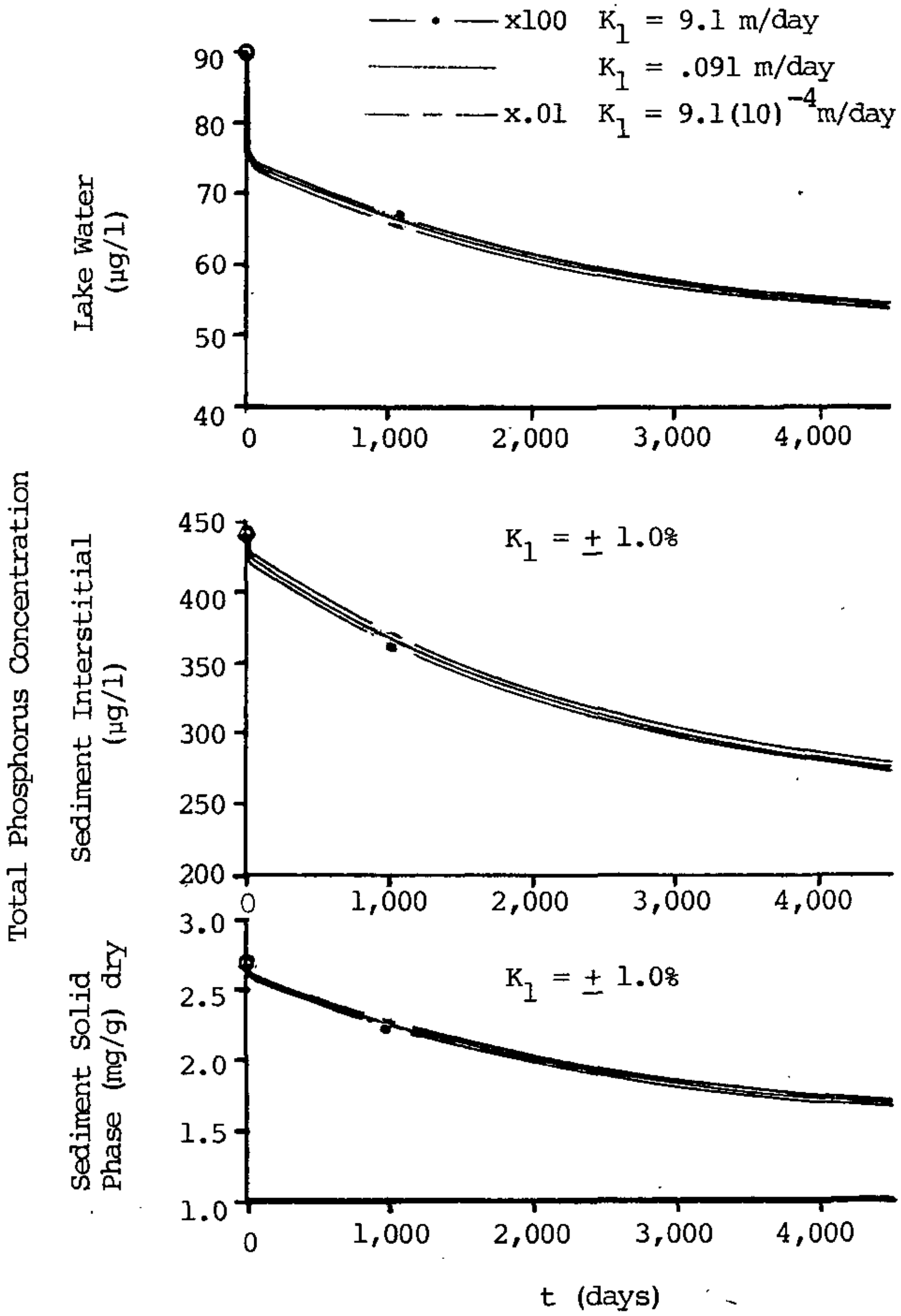


Figure 20. Sensitivity of model to release rate constant K_1.

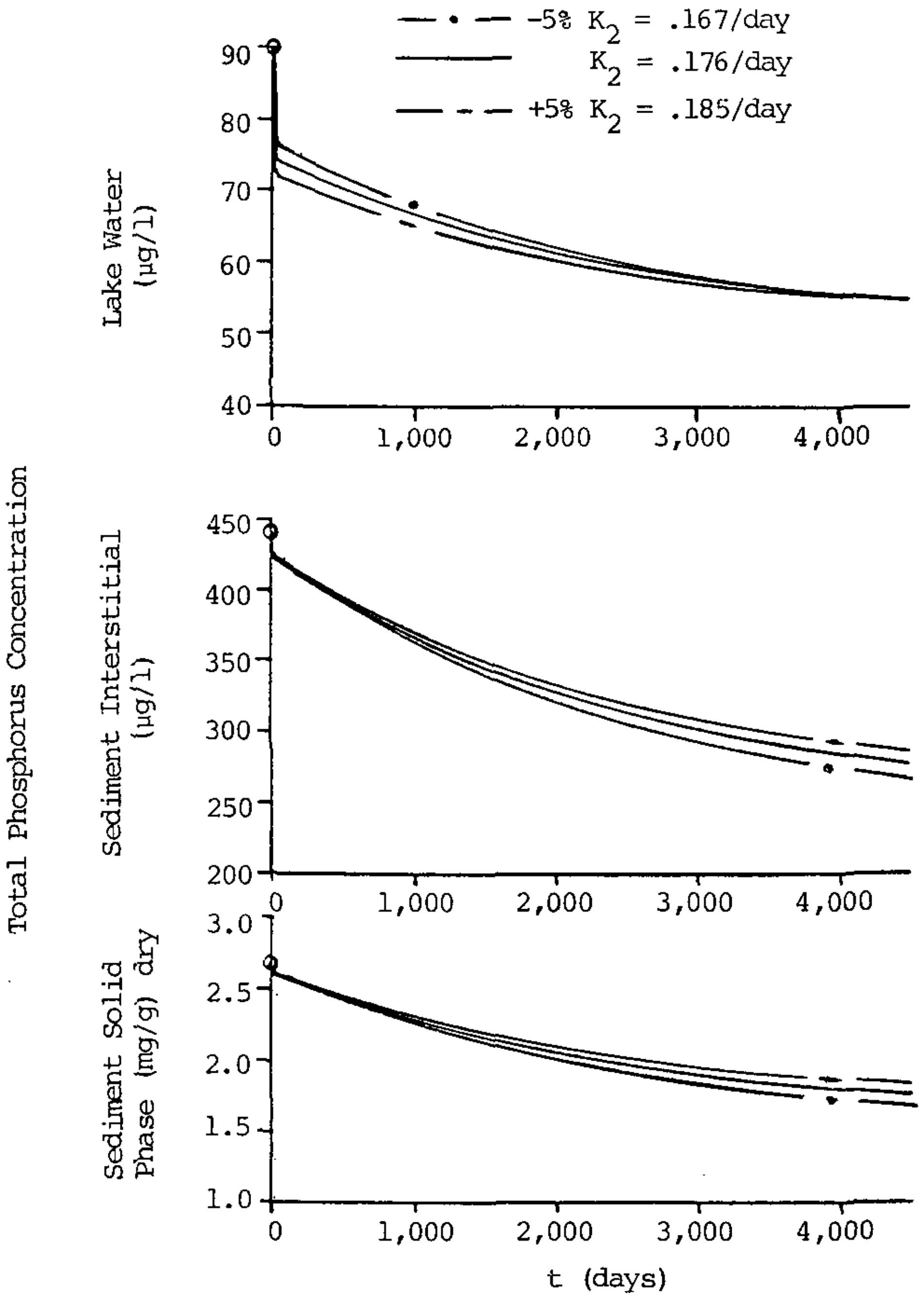


Figure 21. Sensitivity of model to release rate constant K_2.

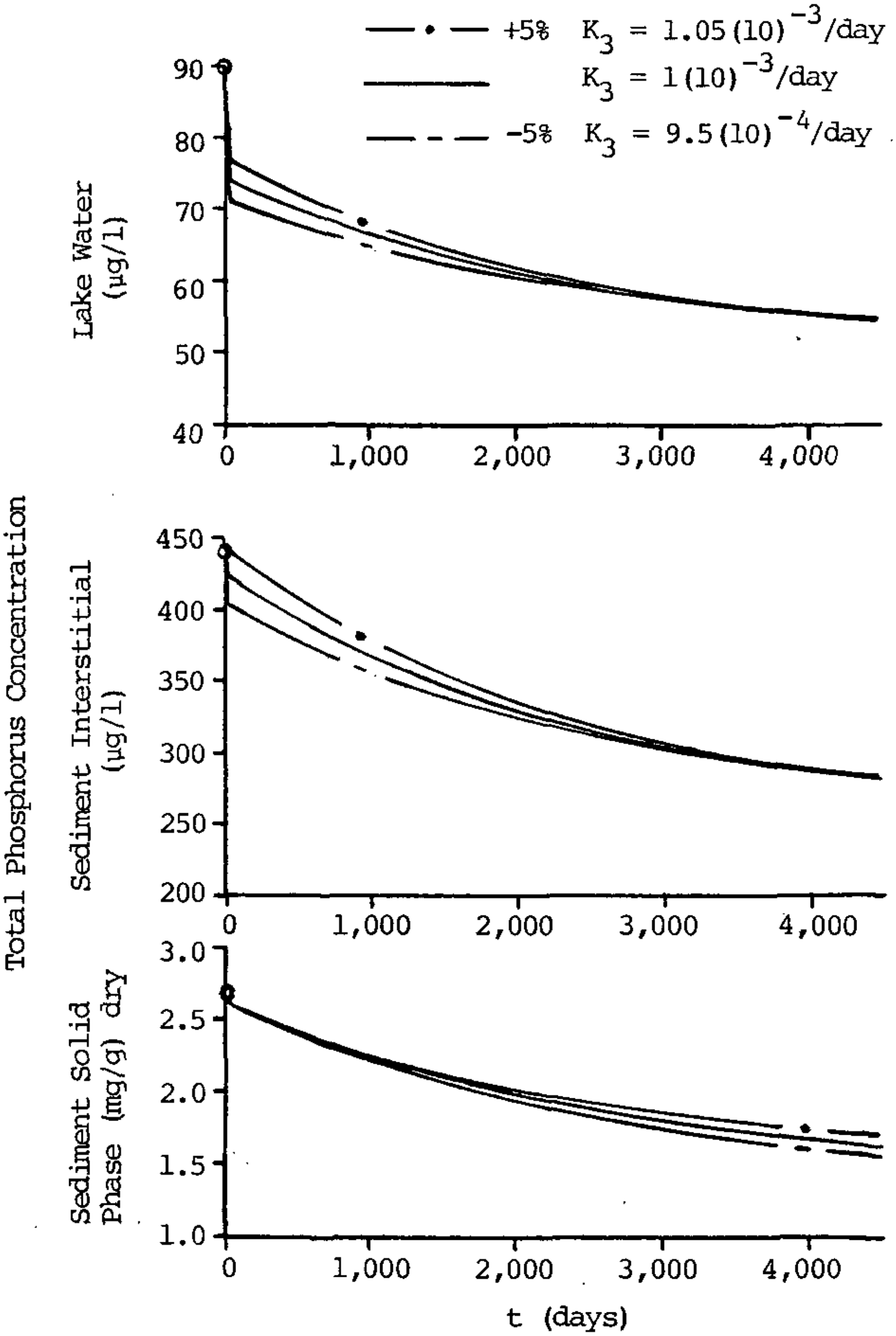


Figure 22. Sensitivity of model predictions to sediment phosphorus desorption rate constant K_3.

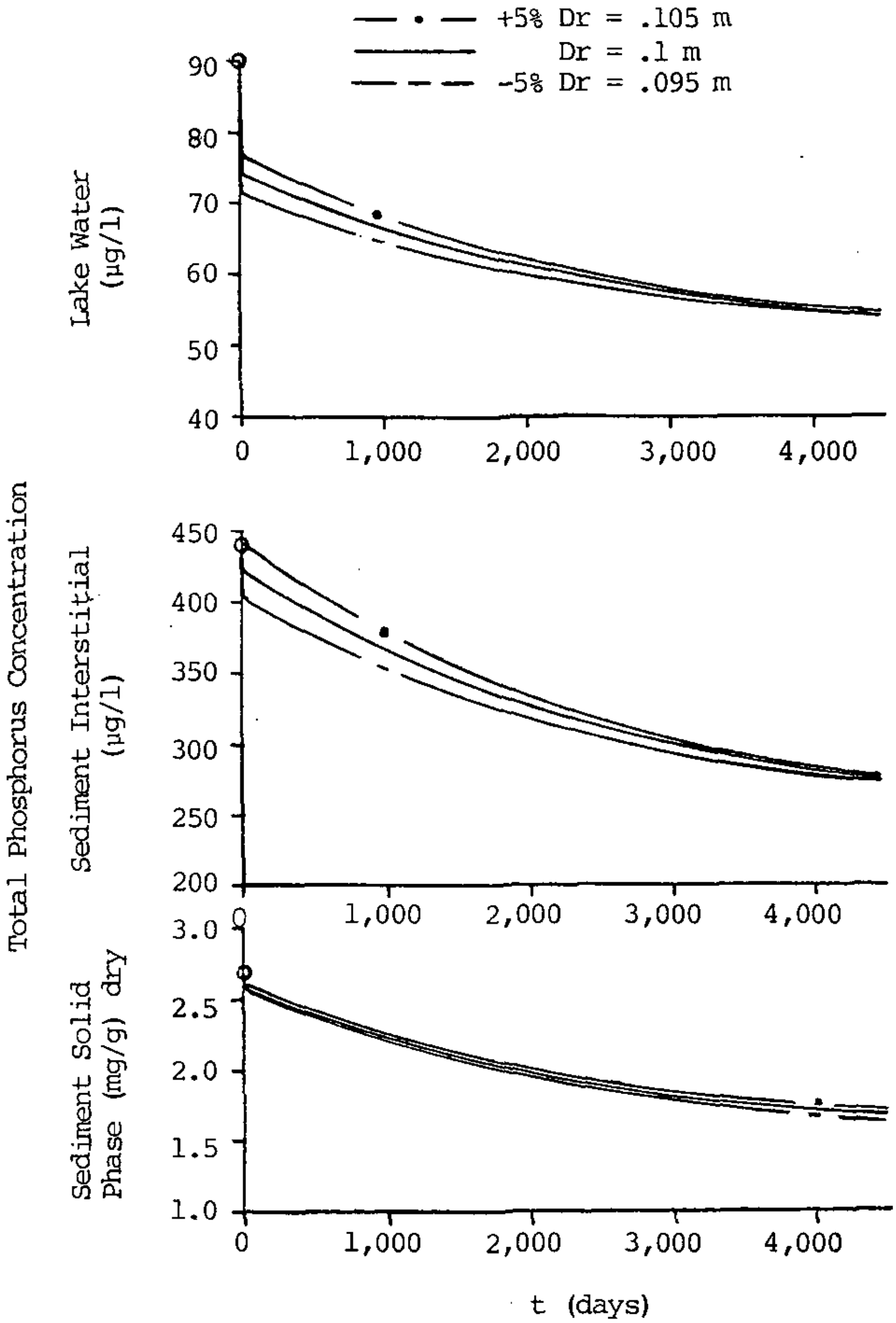


Figure 23. Sensitivity of model to sediment reactive depth Dr.

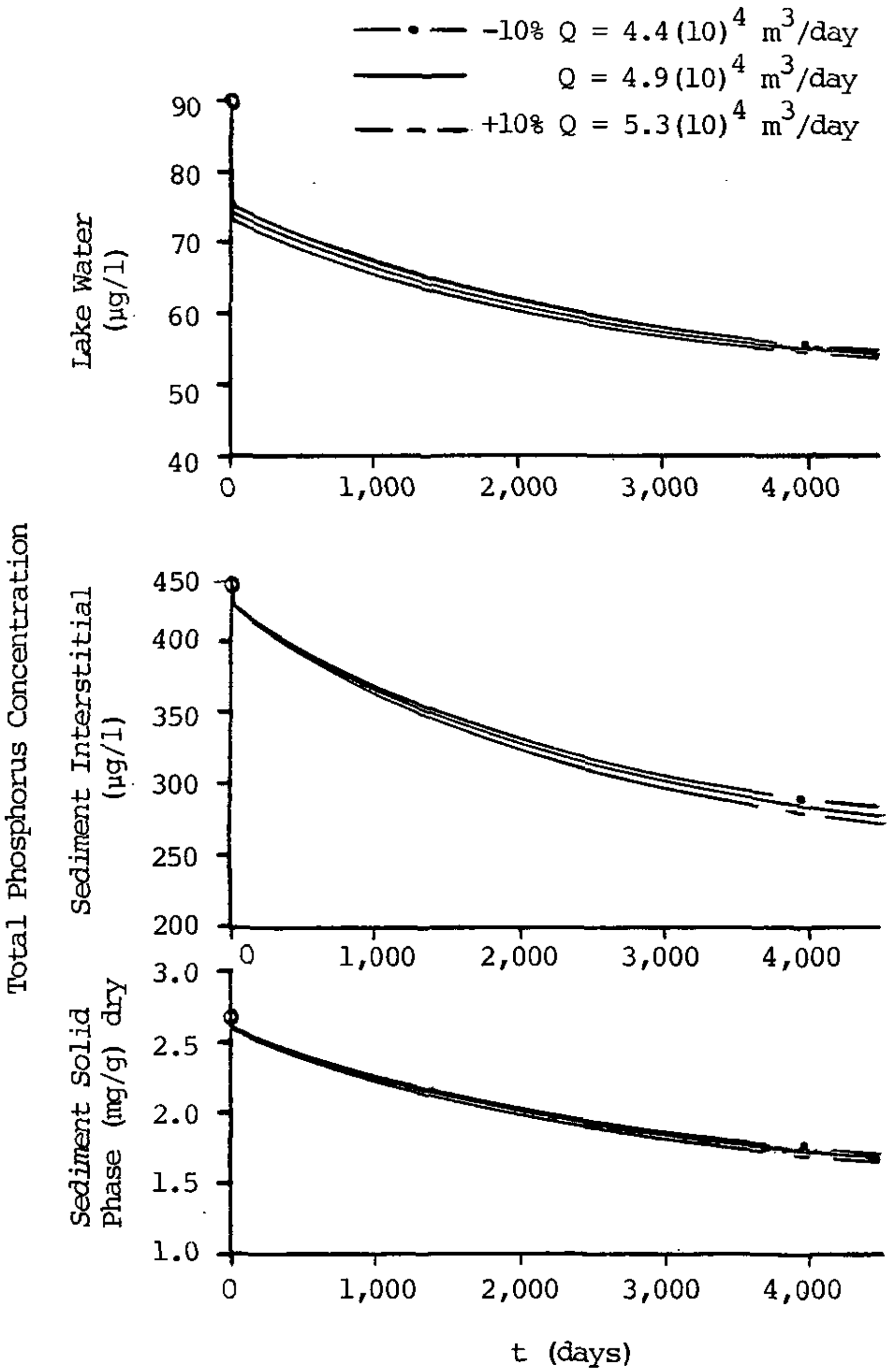


Figure 24. Sensitivity of model to flow rate Q.

Table 3. Values for initial conditions and model parameters used in sensitivity analysis.

ITEM	VALUE
Initial conditions:	
Lake Water Phosphorus Concentration (P_{Lo})	90 μg/l
Sediment Interstitial Water Phosphorus Concentration (P_{io})	440.5 μg/l
Sediment Solid Phase Phosphorus Concentration (P_{so})*	2.6769 (mg/g)dry
Phosphorus Loading ($t > 0$) (P_o)	50 μg/l
K_1	0.091 m/day
K_2	0.176 /day
K_3	10^{-3} /day
Dr	0.1 m
Q	$4.89(10)^4$ m^3/day
V_L	$4.35(10)^5$ m^3
A	$2.572(10)^5$ m^2
	0.84
Exchangeable Sediment Solid Phase Phosphorus (% Exch.)	25.0%

---

*Sediment solid phase phosphorus concentration (P_{so}) is calculated by:

$$P_{so} = P_s(\text{mg/g})\text{dry} \times 10^6 \times (\%\ \text{Exch.})/(\text{sp. Gr.} = 2.5)$$

$t > 5$ years. Long term predictions of more than 5 years of sediment solid phase phosphorus concentrations are sensitive to the value of K_2 and K_3 chosen.

Varying the flow rate (Q) (Figure 24) in the lake model equations produces small changes in predicted phosphorus concentrations. The sensitivity of the lake water equations to Q is constant over time. The sensitivity of the sediment interstitial and sediment solid phase phosphorus concentration to Q increases slightly over time.

Prediction of the dynamic response of the lake water phosphorus concentration from 90 to 70, 50 and 30 (μg/l) are shown in Figure 17. Significant reduction in lake water phosphorus concentration occurs within a few hydraulic detention times due to dilution. The rate of reduction in phosphorus concentration is controlled thereafter by sedimentation and internal phosphorus cycling.

It is expected that the sensitivity to input parameters of the equations obtained by approximation techniques, would be very similar to the sensitivity characteristics shown by the equations obtained by the trial function method.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Phosphate, an essential plant nutrient, is often limiting to plant growth in natural aquatic ecosystems. Total phosphorus is the best index of the biologically available phosphorus. Mass balance models on lake water total phosphorus are useful for predicting the dynamic response of lake phosphorus concentration to pollution control measures. Eutrophic lake bottoms often contain a significant reservoir of phosphorus which may be released to the overlying water and delay the recovery of eutrophic lakes.

Investigations of the hydrogeologic setting of a lake can provide good "first cut" information on the importance of ground water-lake interaction. Seepage meters can be used to measure the magnitude, direction and distribution of ground water interaction with lakes. However, these devices are not considered reliable for sampling seepage water for chemical analysis.

Lake Warner received wastewater overflows from the Amherst Wastewater Treatment Plant for about ten years and began to exhibit eutrophic characteristics. Wastewater overflows were stopped in November, 1975, making Lake Warner a good site for testing mathematical models for the recovery of a eutrophic lake.

Field measurements on the lake phosphorus budget indicated sediment phosphorus release during high water temperature and low flow conditions. A cursory hydrogeologic study of Lake Warner suggested that ground water would discharge into the lake on one side and recharge the ground water aquifer on the other side. Field measurements using seepage meters verified this hypothesis, but showed that ground water was an insignificant component of Lake Warner's water and phosphorus budget and could therefore be neglected in the phosphorus mass balance model. Seepage water samples were not considered representative of sediment pore water chemical composition.

A mass balance on Lake Warner phosphorus indicated that the lake sediments released phosphorus on the July 1981 sampling date and retained phosphorus during September 1981.

The analytical expression for the lake recovery model is a valuable tool for predicting the response of completely mixed lakes, with short water residence times, to changes in phosphorus loading. The implementation of the model for planning, management and research purposes has been facilitated by the development of an analytical expression of the model as opposed to numerical solutions which require the use of high speed computers.

The approximate solutions to the lake model equations are more easily programmed and convenient to use on the TI-59 programmable calculator than are the analytical equations obtained from the trial function method. The short term lake phosphorus equation (equation (4.25)) is very easy to use and can be programmed on calculators with only limited programming capabilities.

The close agreement between the curves obtained by the two solution methods suggests the use of the approximate equations for "first cut" predictions of lake recovery.

Future research on the behavior of conservative materials in deposit-water systems as well as planning and management activities is facilitated by the convenience of the analytical expression of the lake recovery model. Sensitivity analysis on equation (4.21) shows that the value of parameters K_2, K_3 and Dr significantly affect model predictions in the early stages of lake recovery. Care should be taken in estimating the value of these parameters for use in the lake model.

It is recommended that researchers seek to refine the values of reaction rates for particular lakes and that field data used in conjunction with sensitivity analysis be carried out for a wide range of lake systems. As a wider range of lake systems are studied, and more data are accumulated on physical, chemical and biological characteristics of lake bottoms, efforts should be made to compile guidelines which can be employed by model users for choosing the values of reaction rates which would apply to lakes under consideration.

Although beyond the scope of this study it is suggested that efforts be made to relax the assumption of a constant conversion rate (K_3) between sediment solid phase phosphorus and sediment *interstitial phosphorus.* A *time-dependent* K_3 *would, however,* increase the mathematical complexity of the model. Some of the following suggestions might be considered:

i) Use low values of K_3 for the winter season and high values of K_3 for the summer season. Calculate new parameters for each season. Determine the values of the seasonal K_3 values by field investigations.

ii) Concentrate on determining the value of K_3 which would accurately depict sediment conditions in the summer. Sediment phosphorus release during the summer supports biological activity and is, therefore, of interest.

REFERENCES

BINGHAM, D. R. and T. H. FENG, 1980. "Mathematical Modeling of Recovery of a Eutrophic Lake", Report No. Env. E. 65-80-1, Department of Civil Engineering, University of Massachusetts, Amherst, Massachusetts.

BORN, S. M., T. L. WIRTH, J. O. PETERSON, J. P. WALL and D. A. STEPHENSON, 1973. "Dilutional Pumping at Snake Lake, Wisconsin--A Potential Renewal Technique for Small Eutrophic Lakes", Wisconsin Department of Natural Resources, Madison, Wisconsin. Tech. Bull. No. 66.

BROCK, T. D., D. R. LEE, D. JANES and D. WINEK, 1982. "Ground Water Seepage as a Nutrient Source to a Drainage Lake; Lake Mendota, Wisconsin", Water Res. Vol. 16, 1255-1263.

BUCHANAN, T. J. and W. P. SOMERS, 1959. "Discharge Measurements at Gaging Stations", Techniques of Water-Resources Investigations of the United States Geological Survey. Book 3, Chapter A8.

CARTWRIGHT, K., C. S. HUNT, G. M. HUGHES and R. D. BROWER, 1979. "Hydraulic Potential of Lake Michigan Bottom Sediments", Journal of Hydrology, Vol. 403, No. 1/4, 67-78.

CHURCHILL, R. V., 1958. Operational Mathematics. McGraw-Hill Book Company Inc., New York.

CONNOR, J. N. and T. V. BELANGER, 1981. "Ground Water Seepage in Lake Washington and the Upper St. Johns River Basin, Florida", Water Res. Bull. Vol. 17, No. 5, 799-805.

DiGIANO, F. A., 1971. "Mathematical Modeling of Nutrient Transport" in Control of Benthic Deposits in Lakes. T. H. Feng (Ed.) Pub. No. 16. Water Resources Research Center. University of Massachusetts, Amherst, Massachusetts.

DiGIANO, F. A. and P. D. SNOW, 1976. "Consideration of Phosphorus Release from Sediments in a Lake Model", in Interactions between sediments and fresh water. Proceedings of an international symposium held at Amsterdam, the Netherlands, September 6-10, 1976.

DILLON, P. J. 1975. "The Phosphorus Budget of Cameron Lake, Ontario: The Importance of Flushing Rate to the Degree of Eutrophy of Lakes", Limnol. and Oceanogr. Vol. 20, No. 1, 28-39.

DILLON, P. J. and F. H. RIGLER, 1974. "The Phosphorus-Chlorophyll Relationship in Lakes", Limnol. and Oceanogr. Vol. 19, 707-715.

ELSGOLTS, L., 1973. Differential Equations and the Calculus of Variations. Translation by George Yankovsky. MIR Publishers. Moscow.

EMANUEL', N. M. and D. G. KNORRE, 1973. Chemical Kinetics Homogeneous Reactions. Translation from Russian by R. Kondor. John Wiley & Sons, New York.

FELLOWS, C. R. and P. L. BREZONIC, 1980. "Seepage Flow into Florida Lakes", Water Res. Bull. Vol. 16, No. 4, 635-641.

FRAPE, S. K. and R. W. PATTERSON, 1981. "Chemistry of Interstitial Water and Bottom Sediments as Indicators of Seepage Patterns in Perch Lake, Chalk River, Ontario", Limnol. and Oceanogr. Vol. 26, No. 3, 500-517.

FREIFELDER, D., 1982. Physical Chemistry for Students of Biology and Chemistry. Science Books International, Inc. Boston.

HARRIS, G. M., 1966. Chemical Kinetics. D. C. Heath and Company, Boston.

HIMMELBLAU, D. M. and K. B. BISCHOFF, 1968. Process Analysis and Simulation Deterministic Systems. John Wiley and Sons, Inc., New York.

JONES, J. R. and R.W. BACHMANN, 1976. "Prediction of Phosphorus and Chlorophyll Levels in Lakes", J. Water Poll. Contr. Fed. Vol. 48, No. 9, 2176-2182.

JUBINVILLE, R. A., 1973. "Analysis of Phosphorus Budget of Lake Wyola and Lake Warner, Massachusetts", M. S. Special Project, Department of Civil Engineering, University of Massachusetts, Amherst, Massachusetts.

KAPLAN, W., 1957. Advanced Calculus. Addison-Wesley Publishing Company, Inc. Reading, Massachusetts.

KARNAUSKAS, R. J. and M. P. ANDERSON, 1978. "Groundwater Lake Relationships and Groundwater Quality in the Sand Plain Province of Wisconsin-Nepco Lake", Groundwater Vol. 16, No. 4, 273-281.

KEEL, J. K. 1979. "Watershed and Point Source Enrichment and Lake Trophic State Index", EPA-600/3-79-046.

KU, W. C., F. A. DiGIANO and T. H. FENG, 1978. "Factors Affecting Phosphate Adsorption Equilibria in Lake Sediments", Water Research, Vol. 12, 1069-1074.

LAMB, H., 1932. *Hydrodynamics*. *Dover Publications, New York.*

LEAN, D. R. S., 1976. "Movements of Phosphorus Between its Biologically Important Forms in Lake Water", J. Fish. Res. Bd. Canada. Vol. 30, No. 10, 1525-1536.

LEE, D. R., 1976. "The Role of Ground Water in Eutrophication of a Lake in Glacial Outwash Terrain", Int. J. Speleol. Vol. 8, 117-126.

LEE, D. R., J. A. CHERRY and S. F. PICKENS, 1980. "Ground Water Transport of a Salt Tracer through a Sandy Lakebed", Limnol. and Oceanogr. Vol. 25, No. 1, 45-61.

LEE, F. G., W. RAST and R. A. JONES, 1978. "Eutrophication of Waterbodies: Insights into an Age Old Problem", Env. Sci. and Tech. Vol. 12, No. 8, 900-908.

LOCK, M. A. and P. H. JOHN, 1978. "The Measurement of Ground Water Discharge into a Lake by a Direct Method", Int. Revue. Geo. Hydrobiol. Hydrogro. Vol. 63, 271-275.

LOEB, S. L. and G. R. GOLDMAN, 1979. "Water and Nutrient Transport via Ground Water from Ward Valley into Lake Tahoe", Limnol. and Oceanogr. Vol. 24, No. 6, 1146-1154.

LONDQUIST, C. J. 1974. "Contour Map of the Bedrock Surface, Mount Toby Quadrangle, Massachusetts", Miscellaneous Field Studies Map MF-621A. United States Department of the Interior Geological Survey.

LORENZEN, M. W. 1973. "Predicting the Effects of Nutrient Diversion on Lake Recovery", in Modeling the Eutrophication Process Proc. of Workshop at Utah State. Middlebrooks, Falkenborg and Maloney (Eds.) Utah Water Res. Lab., Logan, Utah, NTIS NO. PB22 7065, 205-210.

LORENZEN, M. W., D. J. SMITH and L. V. KIMMEL, 1976. "A Long-Term Phosphorus Model for Lakes: Application to Lake Washington", Modeling Biochemical Processes in Aquatic Ecosystems. Ed. by R. P. Canale, Ann Arbor Science Publishers, Michigan. 75-91.

LUNG, W. and R. P. CANALE, 1977. "Projections of Phosphorus Levels in White Lake", ASCE J. Env. Eng. Div. No. EE4, 663-676.

RINALDI, S., R. SONCINI-SESSA, H. STEHFEST and H. TAMURA, 1979. Modeling and Control of River Quality. McGraw-Hill Inc. Great Britain.

SCHINDLER, D. W., F. A. J. ARMSTRONG, S. K. HOLMGREN and G. J. BRUNSKILL, 1971. "Eutrophication of Lake 227, Experimental Lakes Area, Northwest Ontario, by Addition of Phosphate and Nitrate", J. Fish. Res. Bd. Canada, Vol. 28, 1763-1782.

SNOW, P. D. and F. A. DiGIANO, 1976. "Mathematical Modeling of Phosphorous Exchange between Sediments and Overlying Water in Shallow Eutrophic Lakes", Report No. Env. E. 54-76-3, Department of Civil Engineering, University of Massachusetts, Amherst, Massachusetts.

SNOW, P. D. and F. A. DiGIANO, 1973. "Relationship between Trophic State and Chemical Parameters in Sediment-Water Systems of Selected Western Massachusetts Lakes", Report No. EVE 32-73-3, Department of Civil Engineering, University of Massachusetts, Amherst, Massachusetts.

STRICKLAND, J. D. H. and T. R. PARSONS, 1972. A Practical Handbook of Seawater Analyis, Fisheries Research Board of Canada, Ottawa, Canada.

STUMM, W. and J. O. LECKIE, 1971. "Phosphate Exchange with Sediments; Its Role in the Productivity of Surface Waters." In Proceedings of the Fifth International Conference, Advances in Water Pollution Research, Ed. by S. H. Jenkins, Vol. 2, Article III-26, 1-16.

STUMM, W. and J. J. MORGAN, 1981. Aquatic Chemistry An Introduction Emphasizing Chemical Equilibria in Natural Waters. John Wiley and Sons, Inc., New York.

SYERS, J. K., R. F. HARRIS and D. E. ARMSTRONG, 1973. "Phosphate Chemistry in Lake Sediments", J. Environ. Quality. Vol. 2, No. 1, 1-14.

TUMA, J. J., 1979. Engineering Mathematics Handbook (2nd Ed.), McGraw-Hill Book Company.

VOLLENWEIDER, R. A. 1976. "Advances in Defining Critical Loading Levels for Phosphorus in Lake Eutrophication", Mem. Ist. Ital. Idrobiol. Vol. 33, 53-83.

VOLLENWEIDER, R. A. 1968. "Scientific Fundamentals of the Eutrophication of Lakes and Flowing Waters with Particular Reference to Nitrogen and Phosphorus as Factors in Eutrophication", Tech. Rep. DAS/C81/68,, ORGAN. ECON. COOP. DEV., PARIS.

WILLIAMS, L. R., V. W. LAMBOU, S. C. HERN and R. W. THOMAS, 1978. "Relations of Productivity and Problem Conditions to Ambient Nutrients", EPA-600/3-78-002.

WINTER, T. C. 1978. "Groundwater Component of Lake Water and Nutrient Budgets", Verh. Internat. Verein Limnol. Vol. 20, 438-444.

APPENDIX A

1. MILL RIVER STAGE DISCHARGE CURVES
2. ORTHOPHOSPHATE CALIBRATION CURVE
3. TOTAL PHOSPHORUS CALIBRATION CURVE

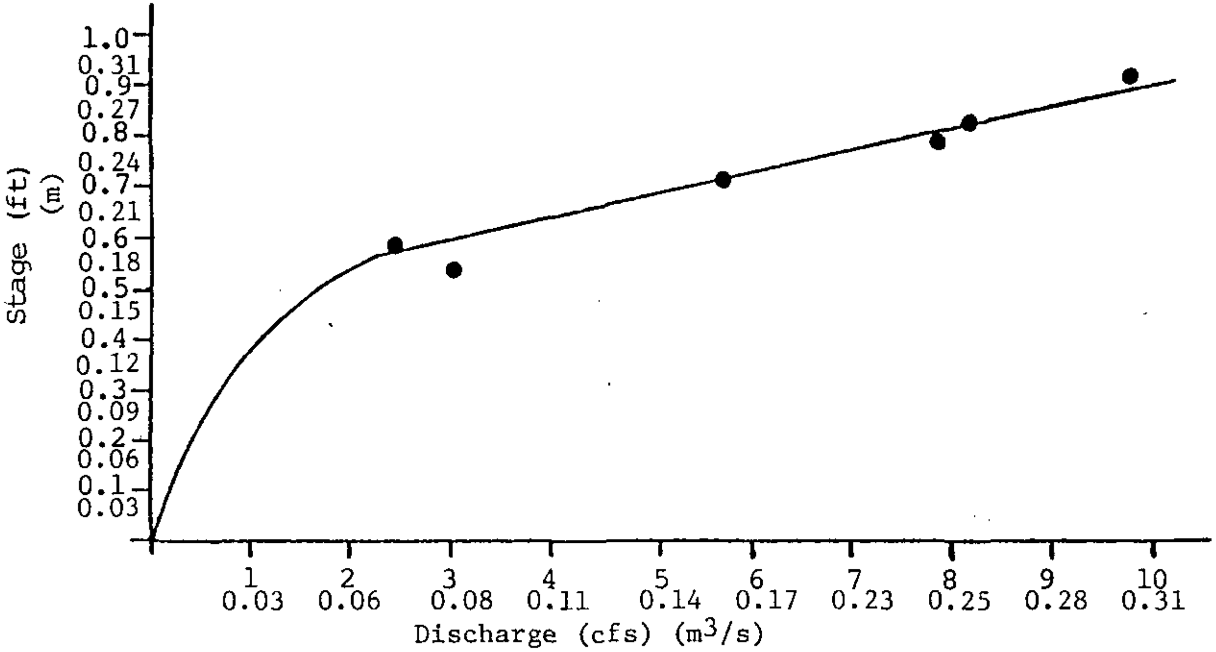


Figure 25. Stage-discharge curve of the Mill River (low flow) at the inlet of Lake Warner.

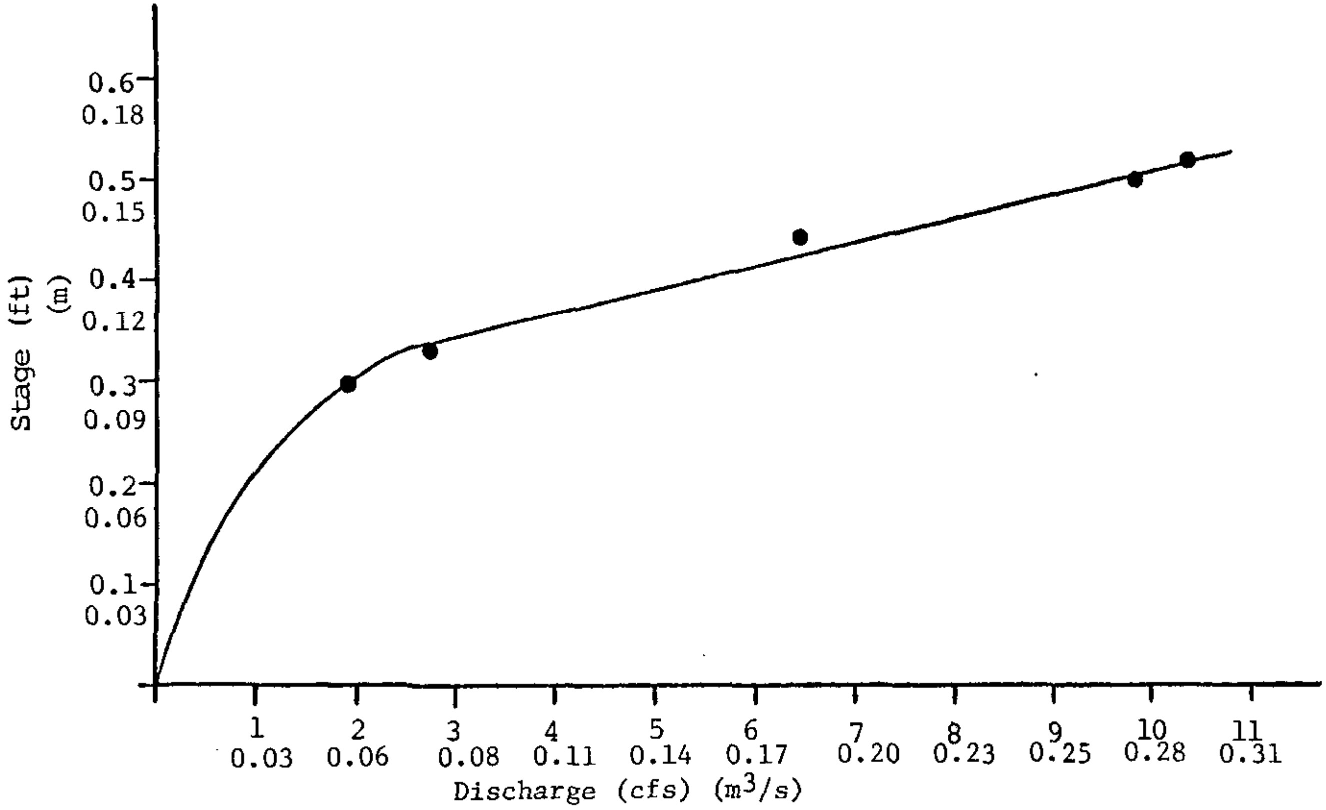


Figure 26. Stage-discharge curve of the Mill River (low flow) at the outlet of Lake Warner.

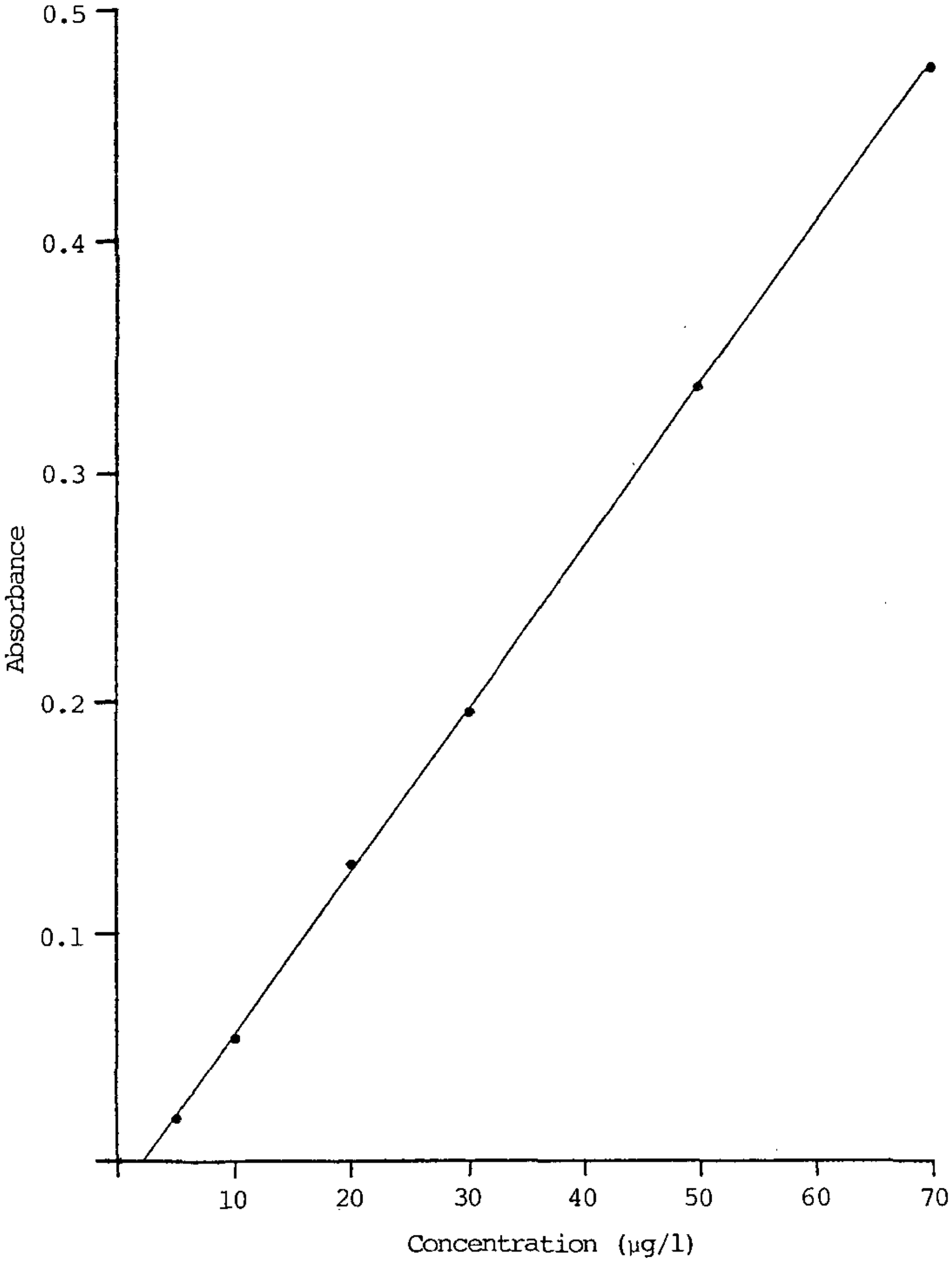


Figure 27. Orthophosphate calibration curve.

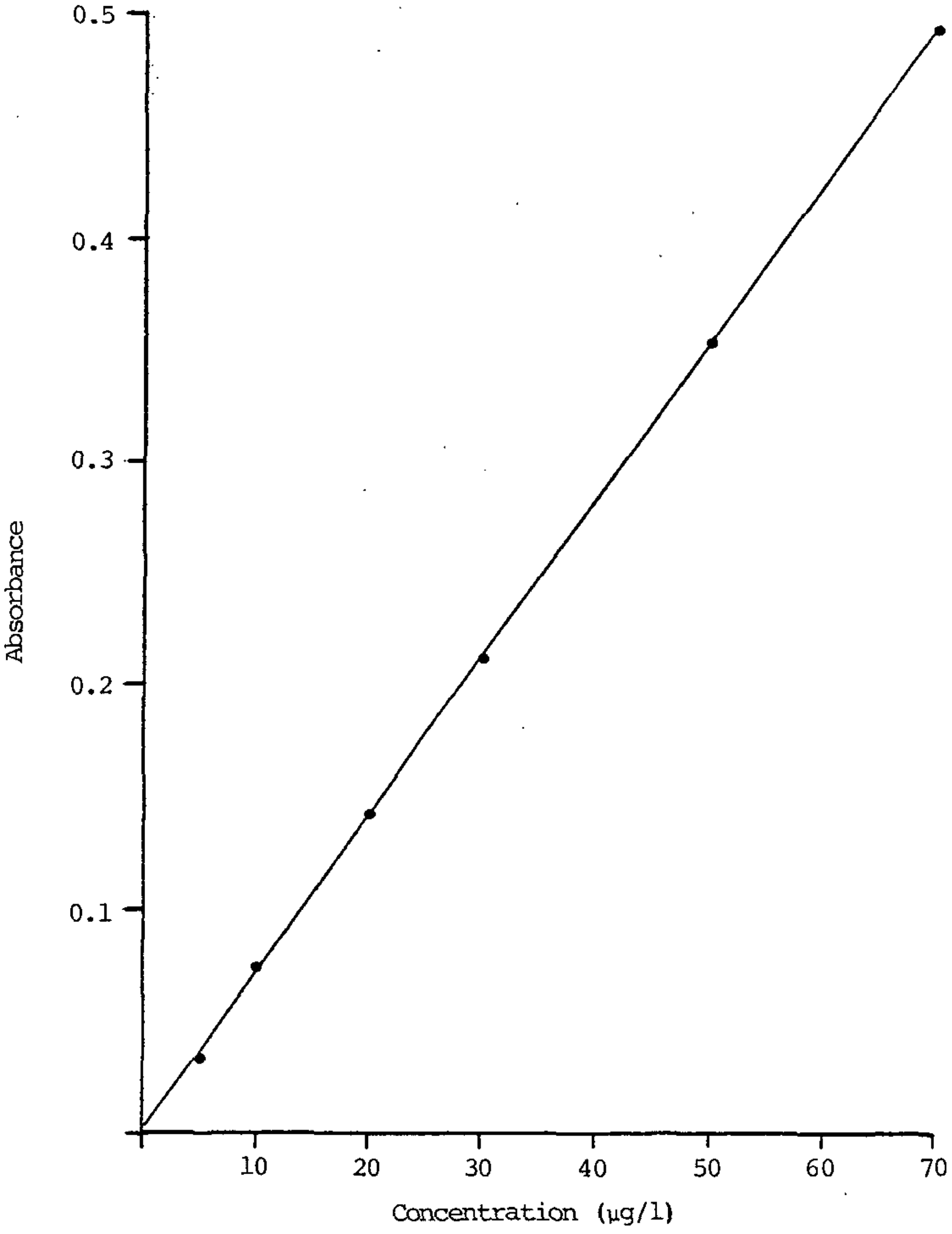


Figure 28. Total phosphorus calibration curve.

APPENDIX B

MATHEMATICAL METHODS

Introduction

The modified Snow and DiGiano (1976) lake recovery model consists of a system of linear first order ordinary differential equations with constant coefficients. Methods for solving such a system of differential equations include:

i) The Laplace transform method

ii) The power series method

iii) The eigenvalue method

iv) The trial function method

v) Numerical methods

iv) Other methods

Snow and DiGiano (1976) and Bingham and Feng (1980) used numerical methods to solve the lake recovery model. The analytical expression for the lake recovery model presented here was developed using the trial function method. In the trial function method the number of homogeneous differential equations (n) to be solved simultaneously is related to the number of algebraic equations (N) needed to solve for the unknown coefficients by equation (7.1):

$$N = n^n + n \qquad 7.1$$

As can be seen from Table 3 the trial function method was a reasonable approach for the solution of the lake recovery model as there are three differential equations to be solved simultaneously. These simultaneous differential equations are probably the maximum which can be conveniently solved by the trial function method.

Table 4. Number of algebraic equations needed to solve for the coefficients in a system of homogeneous ordinary differential equations.

Number of Ordinary Differential Equations n	Number of Simultaneous Algebraic Equations to Solve for the Coefficients N
1	2
2	6
3	12
4	260
5	3,130
6	46,662

The form of the trial function to be postulated for the solution of homogeneous and nonhomogeneous linear first order ordinary differential equations with constant coefficients is discussed below.

<u>Homogeneous case</u>. Given a simple ordinary differential equation:

$$\frac{dX}{dt} = C_1X \tag{7.2}$$

with initial conditions:

$$X(o) = X_o \tag{7.3}$$

a trial function of the form

$$X(t) = me^{\alpha t} \tag{7.4}$$

should be postulated. Taking the derivative of equation (7.4) with respect to t yields:

$$\frac{dX}{dt} = me^{\alpha t} \tag{7.5}$$

substituting (7.4) and (7.5) into equation (7.2) yields:

$$\alpha me^{\alpha t} = C_1 me^{\alpha t} \tag{7.6}$$

Hence, the rate coefficient (α) can be found by solving a linear equation:

$$\alpha = C_1 \tag{7.7}$$

The initial conditions

$$X(o) = X_o, \quad t = 0 \tag{7.3}$$

can be applied to equation (7.4) to yield:

$$X(o) = m \qquad 7.8$$

Substituting equation (7.8) into equation (7.3) yields:

$$m = X_o \qquad 7.9$$

The solution for the homogenous equation (7.2), therefore, is:

$$X(t) = X_o e^{C_1 t} \qquad 7.10$$

which is the same equation that one would obtain by integration of equation (7.2) and the application of the initial conditions.

Similarly one would postulate trial functions of the form:

$$X(t) = m_1 e^{\alpha t} + m_2 e^{\beta t} \qquad 7.11$$

$$Y(t) = n_1 e^{\alpha t} + n_2 e^{\beta t} \qquad 7.12$$

to represent a system of homogeneous ordinary differential equations of the form:

$$\frac{dX}{dt} = C_1 X + C_2 Y \qquad 7.13$$

$$\frac{dY}{dt} = C_3 X + C_4 Y \qquad 7.14$$

Here, the values of α and β would be the roots of a quadratic equation.

Nonhomogeneous case. There are two possible forms of trial function that one could postulate to represent the solution to an ordinary nonhomogeneous differential equation of the form

$$\frac{dX}{dt} = C_1X + C_2 \qquad 7.15$$

with $X = X_o\,, \quad t = 0 \qquad 7.16$

$$X = X_\infty\,, \quad t \longrightarrow \infty \qquad 7.17$$

One form of trial function that one might postulate as a solution to equation (7.15) is:

$$X(t) = m_1(1-e^{\alpha t}) + m_2e^{\delta t} \qquad 7.18$$

where the first term on the right-hand side of equation (7.18) represents the nonhomogeneous portion of the solution to equation (7.15) and the second term allows for a finite initial condition.

Solving the (n) simultaneous algebraic equations to find the coefficients of the trial function that represent (N) ordinary differential equations can be simplified by converting the nonhomogeneous ordinary differential equations to homogeneous ordinary differential equations. This can be done by setting the left-hand side of the ordinary nonhomogeneous equation equal to zero and solving for the dependent variables. A change of variable is then employed such that the nonhomogeneous ordinary differential equation becomes homogeneous. This is the method employed in the solution to the lake recovery model.

The Trial Function Method Applied to the Lake Recovery Model

The modified Snow and DiGiano model presented by Bingham and Feng (1980) to which the trial function method is applied is:

Sediment Solid Phase Phosphorus

$$V_s \frac{dP_s}{dt} = V_L K_3 P_L - V_s K_3 P_s \qquad 4.13$$

Sediment Interstitial Phosphorus

$$\mathcal{E} V_s \frac{dP_i}{dt} = V_s K_3 P_s - \mathcal{E} A K_1 (P_i - P_L) \qquad 4.17$$

Lake Water Phosphorus

$$V_L \frac{dP_L}{dt} = QP_o - QP_L + \mathcal{E} A K_1 (P_i - P_L) - V_L K_2 P_L \qquad 4.9$$

rearranging

$$\frac{dP_s}{dt} = X_1 P_s + X_2 P_L \qquad 4.18$$

$$\frac{dP_i}{dt} = X_3 P_s + X_4 P_i + X_5 P_L \qquad 4.19$$

$$\frac{dP_i}{dt} = X_6 P_i + X_7 P_L + X_8 \qquad 4.20$$

where

$X_1 = -K_3$

$X_2 = K_2V_L/V_s$

$X_3 = K_3/\mathcal{E}$

$X_4 = -AK_1/V_s$

$X_5 = AK_1/V_S$

$X_6 = \mathcal{E}AK_1/V_L$

$X_7 = -(Q + \mathcal{E}AK_1 + K_2V_L)/V_L$

$X_8 = P_oQ/V_L$

These nonhomogeneous ordinary differential equations were converted to homogeneous equations by employing a change of variables so that the function (h) describes the departure from steady state conditions.

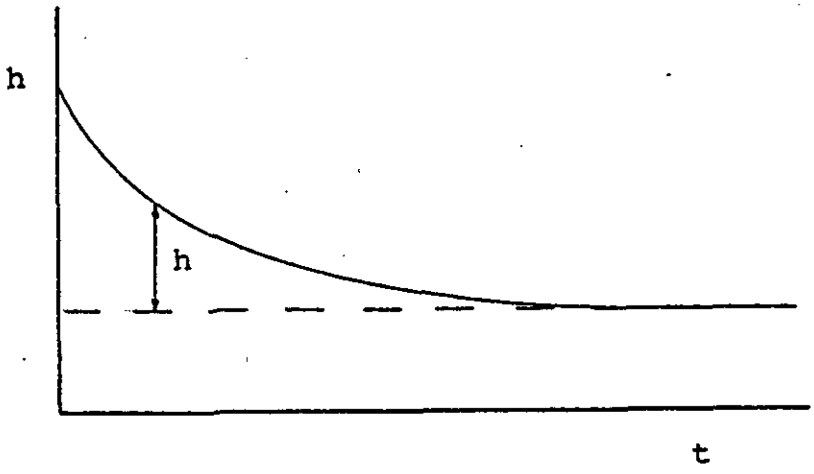


Figure 29. The function h describes the departure from steady state conditions.

$$h_s(t) = P_s(t) - P_{s\infty} \qquad 7.19$$

$$h_i(t) = P_i(t) - P_{i\infty} \qquad 7.20$$

$$h_L(t) = P_L(t) - P_{L\infty} \qquad 7.21$$

Hence, the homogeneous expression for the lake recovery model becomes:

$$\frac{dh_s}{dt} = X_1 h_s + X_2 h_L \qquad 7.22$$

$$\frac{dh_i}{dt} = X_3 h_s + X_4 h_i + X_5 h_L \qquad 7.23$$

$$\frac{dh_L}{dt} = X_6 h_i + X_7 h_L \qquad 7.24$$

The steady state conditions $P_{s\infty}$, $P_{i\infty}$ and $P_{L\infty}$ were found by setting the left hand sides of equations (4.13, 4.17 and 4.9) equal to zero and solving for P_s, P_i and P_L, respectively.

$$P_{s\infty} = K_2 V_L P_o / K_3 V_s \qquad 7.25$$

$$P_{i\infty} = K_2 V_L P_o / \varepsilon K_1 A + P_o \qquad 7.26$$

$$P_{L\infty} = P_o \qquad 7.27$$

After a change of variable has been employed (equations 7.19, 20 and 21) to convert nonhomogeneous equations to homogeneous equations the trial function must satisfy the initial conditions:

$$h = h_o, \quad t = 0$$

Hence, the trial function for the sediment solid phase phosphorus concentration ($h_s(t)$), the sediment interstitial phosphorus concentration ($h_i(t)$) and the lake water phosphorus concentration ($h_L(t)$) can be postulated:

$$h_s(t) = S_1 e^{\alpha t} + S_2 e^{\beta t} + S_3 e^{\gamma t} \qquad 7.28$$

$$h_i(t) = I_1 e^{\alpha t} + I_2 e^{\beta t} + I_3 e^{\gamma t} \qquad 7.29$$

$$h_L(t) = L_1 e^{\alpha t} + L_2 e^{\beta t} + L_3 e^{\gamma t} \qquad 7.30$$

with initial conditions

$$h_s(o) = S_1 + S_2 + S_3 \qquad 7.31$$

$$h_i(o) = I_1 + I_2 + I_3 \qquad 7.32$$

$$h_L(o) = L_1 + L_2 + L_3 \qquad 7.33$$

Taking the derivatives of equations (7.28, 29 and 30) yields:

$$\frac{dh_s}{dt} = \alpha S_1 e^{\alpha t} + \beta S_2 e^{\beta t} + \gamma S_3 e^{\gamma t} \qquad 7.34$$

$$\frac{dh_i}{dt} = \alpha I_1 e^{\alpha t} + \beta I_2 e^{\beta t} + \gamma I_3 e^{\gamma t} \qquad 7.35$$

$$\frac{dh_L}{dt} = \alpha L_1 e^{\alpha t} + \beta L_2 e^{\beta t} + \gamma L_3 e^{\gamma t} \qquad 7.36$$

Applying equations (7.28, 29, 30, 34, 35 and 36) to equations (7.22, 23 and 24) yields:

$$\alpha S_1 e^{\alpha t} + \beta S_2 e^{\beta t} + \gamma S_3 e^{\gamma t} = X_1 \, [S_1 e^{\alpha t} + S_2 e^{\beta t} + S_3 e^{\gamma t}] + X_2 \, [L_1 e^{\alpha t} + L_2 e^{\beta t} + L_3 e^{\gamma t}] \qquad 7.37$$

$$\begin{aligned} &\alpha I_1 e^{\alpha t} + \beta I_2 e^{\beta t} + \gamma I_3 e^{\gamma t} = X_3 \, [S_1 e^{\alpha t} + \\ &S_2 e^{\beta t} + S_3 e^{\gamma t}] + X_4 \, [I_1 e^{\alpha t} + I_2 e^{\beta t} + \\ &I_3 e^{\gamma t}] + X_5 \, [L_1 e^{\alpha t} + L_2 e^{\beta t} + L_3 e^{\gamma t}] \end{aligned} \tag{7.38}$$

$$\begin{aligned} &\alpha L_1 e^{\alpha t} + \beta L_2 e^{\beta t} + \gamma L_3 e^{\gamma t} = X_6 \, [I_1 e^{\alpha t} + \\ &I_2 e^{\beta t} + I_3 e^{\gamma t}] + X_7 \, [L_1 e^{\alpha t} + \\ &L_2 e^{\beta t} + L_3 e^{\gamma t}] \end{aligned} \tag{7.39}$$

Equating the coefficients of $e^{\alpha t}$, $e^{\beta t}$ and $e^{\gamma t}$ in equations (7.37, 38 and 39) yields three sets of equations.

$$\alpha S_1 = X_1 S_1 + X_2 L_1 \tag{7.40.1}$$

$$\alpha I_1 = X_3 S_1 + X_4 I_1 + X_5 L_1 \tag{7.40.2}$$

$$\alpha L_1 = X_6 I_1 + X_7 L_1 \tag{7.40.3}$$

$$\beta S_2 = X_1 S_2 + X_2 L_2 \tag{7.41.1}$$

$$\beta I_2 = X_3 S_2 + X_4 I_2 + X_5 L_2 \tag{7.41.2}$$

$$\beta L_2 = X_6 I_2 + X_7 L_2 \tag{7.41.3}$$

$$\gamma S_3 = X_1 S_3 + X_2 L_3 \tag{7.42.1}$$

$$\gamma I_3 = X_3 S_3 + X_4 I_3 + X_5 L_3 \tag{7.42.2}$$

$$\gamma L_3 = X_6 I_3 + X_7 L_3 \tag{7.42.3}$$

The above yields 9 equations and 12 unknowns. Three more equations that make the solution possible are the initial conditions:

$$h_{so} = S_1 + S_2 + S_3 \qquad 7.31$$

$$h_{io} = I_1 + I_2 + I_3 \qquad 7.32$$

$$h_{Lo} = L_1 + L_2 + L_3 \qquad 7.33$$

Solving for α, β and γ. Each set of equations (7.40.1, 2 and 3), (7.41.1, 2 and 3) and (7.42.1, 2 and 3) can be reduced by substitution, to a cubic equation in α, β and γ, respectively. As an example, equations (7.40.1, 2 and 3) will be solved to arrive at the cubic equation.

Solving for S_1 in equation (7.40.1) yields:

$$S_1 = X_2 L_1/(\alpha - X_1) \qquad 7.43$$

Solving for I_1 equation (7.40.3) yields:

$$I_1 = L_1(\alpha - X_7)/X_6 \qquad 7.44$$

Substituting equations (7.43) and (7.44) into equation (7.40.2) yields a cubic equation in .

$$\alpha^3 - (X_1 + X_7 + X_4)\alpha^2 + (X_1X_7 + X_1X_4 + X_4X_7 - X_5X_6)\alpha + X_1X_5X_6 - X_1X_4X_7 - X_2X_3X_6 = 0 \qquad 4.24$$

Using a similar procedure on equation sets (7.41.1, 2 and 3) and 7.42.1, 2 and 3) identical equations in β and γ can be found.

Hence values for α, β and γ to be used in the lake recovery model are determined to be the roots of equation (4.24).

The roots of equation (4.24) are found using a trigonometric formulation described by Tuma (1979). The general form of equation (4.24) can be expressed as:

$$ar^3 + br^2 + cr + d = 0 \qquad 7.45.1$$

The roots r_1, r_2 and r_3 of equation 7.45.1 are;

$$r_1 = 2\sqrt{|P|/3}\ \cos \phi/3 - b/3a \qquad 7.45.2$$

$$r_2 = -2\sqrt{|P|/3}\ \cos (\phi+\pi)/3 - b/3a \qquad 7.45.3$$

$$r_3 = -2\sqrt{|P|/3}\ \cos (\phi-\pi)/3 - b/3a \qquad 7.45.4$$

with

$$P = [3(c/a) - (b/a)^2]/3 \qquad 7.45.5$$

$$\phi = \cos^{-1} [-q/2\ \sqrt{|P|^3/27}\ ] \qquad 7.45.6$$

where

$$q = [2(b/a)^3 - 9(b/a)(c/a) + 27(d/a)]/27$$

<u>Solving for $P_L(t)$, $P_i(t)$ and $P_s(t)$.</u> Finding the solution to equations $P_L(t)$, $P_i(t)$ and $P_s(t)$ requires the evaluation of the constants L_1, L_2, L_3, I_1, I_2, I_3, and S_1, S_2, S_3 in equations (7.30), (7.29) and (7.28), respectively. This was done by applying equations (7.40.1) through (7.42.3) to the equations defining the initial conditions (equations 7.31, 32 and 33). For

example, to find the constants L_1, L_2 and L_3, equations (7.40.1) through (7.42.3) are employed to find expressions for the constants S_1, S_2, S_3, I_1, I_2 and I_3 in terms of the constants L_1, L_2 and L_3. Hence, for the example of the lake water equations, one obtains:

$$h_{so} = AL_1 + BL_2 + CL_3 \qquad 7.46.1$$

$$h_{io} = DL_1 + EL_2 + FL_3 \qquad 7.46.2$$

$$h_{Lo} = L_1 + L_2 + L_3 \qquad 7.46.3$$

where A through F are defined by equations (4.21.1) through (4.21.6), respectively.

To find $P_L(t)$ the algebraic equations (7.46.1), (7.46.2) and (7.46.3) must be solved simultaneously to find L_1, L_2 and L_3. Substituting the expressions for L_1, L_2 and L_3 into equation (7.30) yields equation (4.21). A similar procedure is used to find the solutions for $P_i(t)$ and $P_s(t)$.

Order of Magnitude Analysis

A common method for simplifying a differential equation is to examine the order of magnitude of the terms in the equation. The order of magnitude of parameters in the lake model equations are:

$P_s = O\ [10^5]\ \mu g/l$ - sediment solid phase phosphorus

$P_i = O\ [10^3]\ \mu g/l$ - sediment interstitial phosphorus

$P_L = O\ [10^2]\ \mu g/l$ - lake water phosphorus

$Dr = O\ [10^{-1}]\ m$ - sediment reactive depth

$Z = O\ [1]\ m$ - lake mean depth

$T = O\ [10]$ days - lake detention time

$K_1 = O\ [10^{-1}]\ m/day$ - sediment release rate

$K_2 = O\ [10^{-1}]\ /day$ - sedimentation rate

$K_3 = O\ [10^{-3}]\ /day$ - sediment desorption rate

$\mathcal{E} = O\ [1]$ - sediment porosity

Order of magnitude of the terms in equation (4.18).

$$O\ [K_3P_s] \quad = \quad O\ [10^2]$$

$$O\ [ZK_2P_L/D_r] \quad = \quad O\ [10^2)$$

Hence:

$$O\ [ZK_2P_L/Dr] \quad = \quad O\ [K_3P_s]$$

Order of magnitude of the terms in equation (4.19).

$$O\ [K_3P_s/\mathcal{E}\ ] \quad = \quad O\ [10^2]$$

$$O\ [K_1P_i/Dr] \quad = \quad O\ [10^3]$$

$$O\ [K_1P_L/Dr] \quad = \quad O\ [10^2]$$

Hence:

$$K_3P_s/\mathcal{E} \quad , \quad K_1P_L/Dr \ll K_1P_i/Dr$$

Order of magnitude of terms in equation (4.20).

$$O\,[P_L/T] = O\,[10]$$

$$O\,[\,\mathcal{E} K_1 P_i/Z] = O\,[10^2]$$

$$O\,[\,\mathcal{E} K_1 P_L/Z] = O\,[10]$$

Hence:

$$P_L/T \quad , \quad \mathcal{E} K_1 P_L/Z \ll \mathcal{E} K_1 P_i/Z$$

Order of magnitude analysis indicates that the order of P_i is greater than the other terms in equation (4.20). This fact allowed the simplification of the lake water equation for intermediate term predictions.

APPENDIX C

CODING FOR LAKE MODEL EQUATIONS

(TEXAS INSTRUMENTS 59 PROGRAMMABLE CALCULATOR)

PARAMETERS

This program calculates the parameters (X_1, X_2, X_3, X_4, X_5, X_6, X_7, α, β, γ) and values of P_{io}, P_{so}, $P_{i\infty}$ and $P_{s\infty}$ and stores them in data registers 16, 17, 19 and 20, respectively, for use in the lake recovery model equations $P_L(t)$, $P_i(t)$ and $P_s(t)$.

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter constants	K_1	STO21	K_1
	K_2	STO22	K_2
	$-K_3$	STO23	$-K_3$
	ε	STO24	ε
	D_r	STO25	D_r
	Q	STO26	Q
	V_L	STO27	V_L
	A	STO28	A
	P_o	STO29	P_o
	P_{Lo}	STO15	P_{Lo}
2. Run Program		E	α
3. Continue Program		R/S	β
4. Continue Program		R/S	γ

Parameter	DATA REGISTER	Parameter	DATA REGISTER
input:			
K_1 m/day	21	X_3 /day	06
K_2 /day	22	X_4 /day	07
$-K_3$ /day	23	X_5 /day	08
$\mathcal{E}$	24	X_6 /day	09
Dr meters	25	X_7 /day	10
Q m^3/day	26	P	30
V_L m^3	27	Q	31
A m^2	28	ϕ	32
P_o μg/l	29, 18	b	34
P_{Lo} μg/l	15	c	35
t days	00	d	36
calculated by parameter program:			
P_{io} μg/l	16		
P_{so} μg/l	17		
$P_{L\infty}$ μg/l	18		
$P_{i\infty}$ μg/l	19		
$P_{s\infty}$ μg/l	20		
α /day	01		
β /day	02		
γ /day	03		
X_1 /day	04		
X_2 /day	05		

```
000  76 LBL
001  11  A
002  43 RCL
003  29  29
004  42 STO
005  18  18
006  53  (
007  43 RCL
008  22  22
009  65  ×
010  43 RCL
011  27  27
012  55  ÷
013  43 RCL
014  28  28
015  55  ÷
016  43 RCL
017  25  25
018  54  )
019  95  =
020  42 STO
021  05  05
022  43 RCL
023  23  23
024  42 STO
025  04  04
026  53  (
027  43 RCL
028  23  23
029  55  ÷
030  43 RCL
031  24  24
032  94 +/-
033  54  )
034  95  =
035  42 STO
036  06  06
037  53  (
038  43 RCL
039  28  28
040  65  ×
041  43 RCL
042  21  21
043  55  ÷
044  43 RCL
045  28  28
046  55  ÷
047  43 RCL
048  25  25
049  54  )
050  95  =
051  42 STO
052  08  08
053  94 +/-
054  42 STO
055  07  07
056  53  (
057  43 RCL
058  24  24
059  65  ×
060  43 RCL
061  28  28
062  65  ×
063  43 RCL
064  21  21
065  55  ÷
066  43 RCL
067  27  27
068  54  )
069  95  =
070  42 STO
071  09  09
072  53  (
073  53  (
074  43 RCL
075  26  26
076  85  +
077  43 RCL
078  24  24
079  65  ×
080  43 RCL
081  28  28
082  65  ×
083  43 RCL
084  21  21
085  85  +
086  43 RCL
087  22  22
088  65  ×
089  43 RCL
090  27  27
091  54  )
092  55  ÷
093  43 RCL
094  27  27
095  54  )
096  95  =
097  94 +/-
098  42 STO
099  10  10
100  53  (
101  43 RCL
102  05  05
103  55  ÷
104  43 RCL
105  08  08
106  55  ÷
107  43 RCL
108  24  24
109  85  +
110  01  1
111  54  )
112  65  ×
113  43 RCL
114  15  15
115  95  =
116  42 STO
117  16  16
118  53  (
119  43 RCL
120  05  05
121  55  ÷
122  43 RCL
123  08  08
124  55  ÷
125  43 RCL
126  24  24
127  85  +
128  01  1
129  54  )
130  65  ×
131  43 RCL
132  29  29
133  95  =
134  42 STO
135  19  19
136  43 RCL
137  05  05
138  55  ÷
139  43 RCL
140  23  23
141  65  ×
142  43 RCL
143  15  15
144  94 +/-
145  95  =
146  42 STO
147  17  17
148  43 RCL
149  05  05
150  55  ÷
151  43 RCL
152  23  23
153  94 +/-
154  65  ×
155  43 RCL
156  18  18
157  95  =
158  42 STO
159  20  20
160  92 RTN
161  76 LBL
162  12  B
163  53  (
164  43 RCL
165  04  04
166  85  +
167  43 RCL
168  07  07
169  85  +
170  43 RCL
171  10  10
172  54  )
173  94 +/-
174  95  =
175  42 STO
176  34  34
177  53  (
178  43 RCL
179  04  04
180  65  ×
181  43 RCL
182  07  07
183  85  +
184  43 RCL
185  04  04
186  65  ×
187  43 RCL
188  10  10
189  85  +
190  43 RCL
191  07  07
192  65  ×
193  43 RCL
194  10  10
195  75  -
196  43 RCL
197  08  08
198  65  ×
199  43 RCL
200  09  09
201  54  )
202  95  =
203  42 STO
204  35  35
205  53  (
206  43 RCL
207  04  04
208  65  ×
209  43 RCL
210  08  08
211  65  ×
212  43 RCL
213  09  09
214  75  -
215  43 RCL
216  04  04
217  65  ×
218  43 RCL
219  07  07
220  65  ×
221  43 RCL
222  10  10
223  75  -
224  43 RCL
225  05  05
226  65  ×
227  43 RCL
228  06  06
229  65  ×
230  43 RCL
231  09  09
232  54  )
233  95  =
234  42 STO
235  36  36
236  92 RTN
237  76 LBL
238  13  C
239  53  (
240  43 RCL
241  35  35
242  65  ×
243  03  3
244  75  -
245  43 RCL
246  34  34
247  33 X²
248  54  )
249  55  ÷
250  03  3
251  95  =
252  42 STO
253  30  30
254  53  (
255  53  (
256  43 RCL
257  34  34
258  45 Yˣ
259  03  3
260  54  )
261  65  ×
262  02  2
263  75  -
264  43 RCL
265  34  34
266  65  ×
267  43 RCL
268  35  35
269  65  ×
270  09  9
271  85  +
272  43 RCL
273  36  36
274  65  ×
275  02  2
276  07  7
277  54  )
278  55  ÷
279  02  2
280  07  7
281  95  =
282  42 STO
283  31  31
284  92 RTN
285  76 LBL
286  14  D
287  53  (
288  53  (
289  53  (
290  43 RCL
291  30  30
292  94 +/-
293  45 Yˣ
294  03  3
295  54  )
296  55  ÷
297  02  2
298  07  7
299  54  )
```

Figure 30. Coding for TI-59 Programmable Calculator, Parameters Program.

```
300  34 √X     360  53  (             0.          00
301  35 1/X    361  43 RCL   -.0003886519         01
302  65  ×     362  32  32   -.2702561925         02
303  43 RCL    363  85  +    -.9739707142         03
304  31  31    364  89  π          -0.001         04
305  94 +/-    365  54  )     2.976671851         05
306  55  ÷     366  55  ÷     .0011904762         06
307  02  2     367  03  3           -0.91         07
308  54  )     368  54  )            0.91         08
309  70 RAD    369  70 RAD    .0451962483         09
310  22 INV    370  39 COS   -.3336155586         10
311  39 COS    371  65  ×              0.         11
312  95  =     372  43 RCL             0.         12
313  42 STO    373  33  33             0.         13
314  32  32    374  94 +/-             0.         14
315  53  (     375  75  -             90.         15
316  43 RCL    376  53  (      440.471568         16
317  30  30    377  43 RCL    267900.4666         17
318  94 +/-    378  34  34            50.         18
319  55  ÷     379  55  ÷     244.7064267         19
320  03  3     380  03  3     148833.5925         20
321  54  )     381  54  )           0.091         21
322  34 √X     382  95  =           0.176         22
323  65  ×     383  42 STO         -0.001         23
324  02  2     384  02  02           0.84         24
325  95  =     385  91 R/S            0.1         25
326  42 STO    386  53  (         48902.4         26
327  33  33    387  53  (         435000.         27
328  92 RTN    388  43 RCL        257200.         28
329  76 LBL    389  32  32            50.         29
330  15  E     390  75  -    -.2526507749         30
331  11  A     391  89  π     .0335129117         31
332  12  B     392  54  )     2.326247743         32
333  13  C     393  55  ÷     .5804030496         33
334  14  D     394  03  3     1.244615559         34
335  53  (     395  54  )     0.263705188         35
336  53  (     396  70 RAD    .0001023016         36
337  43 RCL    397  39 COS             0.         37
338  32  32    398  65  ×              0.         38
339  55  ÷     399  43 RCL             0.         39
340  03  3     400  33  33
341  54  )     401  94 +/-
342  70 RAD    402  75  -
343  39 COS    403  53  (
344  65  ×     404  43 RCL
345  43 RCL    405  34  34
346  33  33    406  55  ÷
347  54  )     407  03  3
348  75  -     408  54  )
349  53  (     409  95  =
350  43 RCL    410  42 STO
351  34  34    411  03  03
352  55  ÷     412  91 R/S
353  03  3
354  54  )
355  95  =
356  42 STO
357  01  01
358  91 R/S
359  53  (
```

Figure 30. (continued)

LAKE WATER PHOSPHORUS CONCENTRATION

The following program calculates the lake water phosphorus concentration ($P_L(t)$) (equation (4.21)) in response to a change in phosphorus inflow concentration ($P_o = P_{Lo}$, $t < 0$; $P_o = P_{L\infty}$, $t \geqslant 0$). The parameters X_1, X_2, X_3, X_4, X_5, X_6, X_6, X_7, α, β and γ from the parameters program must be used.

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter time (days)	t	STO00	t
2. Run program		E	$P_L(t)$
3. Enter time (days)	t	STO00	t
4. Run program		R/S	$P_L(t)$

```
000  76 LBL    060  09  09    120  09  09    180  55  ÷     240  04  04
001  16 A'     061  54  )     121  54  )     181  53  (     241  54  )
002  53  (     062  55  ÷     122  42 STO    182  43 RCL    242  54  )
003  53  (     063  43 RCL    123  24  24    183  02  02    243  42 STO
004  53  (     064  06  06    124  92 RTN    184  75  -     244  26  26
005  43 RCL    065  55  ÷     125  55  ÷     185  43 RCL    245  92 RTN
006  01  01    066  43 RCL    126  53  (     186  07  07    246  76 LBL
007  75  -     067  09  09    127  43 RCL    187  54  )     247  38 SIN
008  43 RCL    068  54  )     128  01  01    188  55  ÷     248  53  (
009  07  07    069  42 STO    129  75  -     189  53  (     249  53  (
010  54  )     070  22  22    130  43 RCL    190  43 RCL    250  43 RCL
011  65  ×     071  92 RTN    131  04  04    191  02  02    251  25  25
012  53  (     072  76 LBL    132  54  )     192  75  -     252  75  -
013  43 RCL    073  18 C'     133  54  )     193  43 RCL    253  43 RCL
014  01  01    074  53  (     134  55  ÷     194  04  04    254  24  24
015  75  -     075  53  (     135  53  (     195  54  )     255  54  )
016  43 RCL    076  53  (     136  43 RCL    196  54  )     256  65  ×
017  10  10    077  43 RCL    137  01  01    197  42 STO    257  53  (
018  54  )     078  03  03    138  75  -     198  25  25    258  43 RCL
019  75  -     079  75  -     139  43 RCL    199  92 RTN    259  22  22
020  43 RCL    080  43 RCL    140  07  07    200  76 LBL    260  65  ×
021  08  08    081  07  07    141  54  )     201  11  A     261  43 RCL
022  65  ×     082  54  )     142  55  ÷     202  53  (     262  26  26
023  43 RCL    083  65  ×     143  53  (     203  53  (     263  75  -
024  09  09    084  53  (     144  43 RCL    204  43 RCL    264  43 RCL
025  54  )     085  43 RCL    145  01  01    205  03  03    265  23  23
026  55  ÷     086  03  03    146  75  -     206  75  -     266  65  ×
027  43 RCL    087  75  -     147  43 RCL    207  43 RCL    267  43 RCL
028  06  06    088  43 RCL    148  04  04    208  04  04    268  25  25
029  55  ÷     089  10  10    149  54  )     209  54  )     269  54  )
030  43 RCL    090  54  )     150  54  )     210  55  ÷     270  75  -
031  09  09    091  75  -     151  42 STO    211  43 RCL    271  53  (
032  54  )     092  43 RCL    152  24  24    212  09  09    272  43 RCL
033  42 STO    093  08  08    153  92 RTN    213  54  )     273  25  25
034  21  21    094  65  ×     154  76 LBL    214  42 STO    274  75  -
035  92 RTN    095  43 RCL    155  10 E'     215  26  26    275  43 RCL
036  76 LBL    096  09  09    156  53  (     216  92 RTN    276  26  26
037  17 B'     097  54  )     157  53  (     217  65  ×     277  54  )
038  53  (     098  55  ÷     158  43 RCL    218  53  (     278  65  ×
039  53  (     099  43 RCL    159  02  02    219  43 RCL    279  53  (
040  53  (     100  06  06    160  75  -     220  03  03    280  43 RCL
041  43 RCL    101  55  ÷     161  43 RCL    221  75  -     281  22  22
042  02  02    102  43 RCL    162  04  04    222  43 RCL    282  65  ×
043  75  -     103  09  09    163  54  )     223  04  04    283  43 RCL
044  43 RCL    104  54  )     164  55  ÷     224  54  )     284  24  24
045  07  07    105  42 STO    165  43 RCL    225  54  )     285  75  -
046  54  )     106  23  23    166  09  09    226  55  ÷     286  43 RCL
047  65  ×     107  92 RTN    167  54  )     227  53  (     287  21  21
048  53  (     108  76 LBL    168  42 STO    228  43 RCL    288  65  ×
049  43 RCL    109  19 D'     169  25  25    229  03  03    289  43 RCL
050  02  02    110  53  (     170  92 RTN    230  75  -     290  25  25
051  75  -     111  53  (     171  65  ×     231  43 RCL    291  54  )
052  43 RCL    112  43 RCL    172  53  (     232  07  07    292  54  )
053  10  10    113  01  01    173  43 RCL    233  54  )     293  95  =
054  54  )     114  75  -     174  02  02    234  55  ÷     294  42 STO
055  75  -     115  43 RCL    175  75  -     235  53  (     295  27  27
056  43 RCL    116  04  04    176  43 RCL    236  43 RCL    296  53  (
057  08  08    117  54  )     177  04  04    237  03  03    297  53  (
058  65  ×     118  55  ÷     178  54  )     238  75  -     298  43 RCL
059  43 RCL    119  43 RCL    179  54  )     239  43 RCL    299  25  25
```

Figure 31. Coding for TI-59 Programmable Calculator, lake model equation $P_L(t)$ (4.21) solved by Trial Function Method.

```
300  75  -     360  43 RCL    420  22  22    480  43 RCL    540  54  )
301  43 RCL    361  22  22    421  65  ×     481  21  21    541  22 INV
302  25  25    362  65  ×     422  43 RCL    482  65  ×     542  23 LNX
303  54  )     363  43 RCL    423  26  26    483  43 RCL    543  65  ×
304  65  ×     364  24  24    424  75  -     484  25  25    544  43 RCL
305  53  (     365  54  )     425  43 RCL    485  75  -     545  26  26
306  43 RCL    366  75  -     426  23  23    486  43 RCL    546  65  ×
307  23  23    367  53  (     427  65  ×     487  24  24    547  53  (
308  65  ×     368  43 RCL    428  43 RCL    488  65  ×     548  43 RCL
309  43 RCL    369  24  24    429  25  25    489  43 RCL    549  26  26
310  24  24    370  75  -     430  54  )     490  22  22    550  75  -
311  75  -     371  43 RCL    431  55  ÷     491  54  )     551  43 RCL
312  43 RCL    372  25  25    432  43 RCL    492  55  ÷     552  24  24
313  21  21    373  54  )     433  27  27    493  43 RCL    553  54  )
314  65  ×     374  65  ×     434  85  +     494  29  29    554  55  ÷
315  43 RCL    375  53  (     435  53  (     495  54  )     555  43 RCL
316  26  26    376  43 RCL    436  43 RCL    496  54  )     556  28  28
317  54  )     377  21  21    437  02  02    497  95  =     557  85  +
318  75  -     378  65  ×     438  65  ×     498  92 RTN    558  53  (
319  53  (     379  43 RCL    439  43 RCL    499  76 LBL    559  53  (
320  43 RCL    380  26  26    440  00  00    500  13  C     560  43 RCL
321  26  26    381  75  -     441  54  )     501  53  (     561  03  03
322  75  -     382  43 RCL    442  22 INV    502  43 RCL    562  65  ×
323  43 RCL    383  23  23    443  23 LNX    503  17  17    563  43 RCL
324  24  24    384  65  ×     444  65  ×     504  75  -     564  00  00
325  54  )     385  43 RCL    445  43 RCL    505  43 RCL    565  54  )
326  65  ×     386  24  24    446  26  26    506  20  20    566  22 INV
327  53  (     387  54  )     447  65  ×     507  54  )     567  23 LNX
328  43 RCL    388  54  )     448  53  (     508  65  ×     568  65  ×
329  23  23    389  95  =     449  43 RCL    509  53  (     569  43 RCL
330  65  ×     390  42 STO    450  23  23    510  53  (     570  24  24
331  43 RCL    391  29  29    451  65  ×     511  43 RCL    571  65  ×
332  25  25    392  61 GTO    452  43 RCL    512  01  01    572  53  (
333  75  -     393  39 COS    453  24  24    513  65  ×     573  43 RCL
334  43 RCL    394  76 LBL    454  75  -     514  43 RCL    574  24  24
335  22  22    395  12  B     455  43 RCL    515  00  00    575  75  -
336  65  ×     396  53  (     456  21  21    516  54  )     576  43 RCL
337  43 RCL    397  43 RCL    457  65  ×     517  22 INV    577  25  25
338  26  26    398  15  15    458  43 RCL    518  23 LNX    578  54  )
339  54  )     399  75  -     459  26  26    519  65  ×     579  55  ÷
340  54  )     400  43 RCL    460  54  )     520  43 RCL    580  43 RCL
341  95  =     401  18  18    461  55  ÷     521  25  25    581  29  29
342  42 STO    402  54  )     462  43 RCL    522  65  ×     582  54  )
343  28  28    403  65  ×     463  28  28    523  53  (     583  54  )
344  53  (     404  53  (     464  85  +     524  43 RCL    584  95  =
345  53  (     405  53  (     465  53  (     525  25  25    585  92 RTN
346  43 RCL    406  43 RCL    466  53  (     526  75  -     586  76 LBL
347  24  24    407  01  01    467  43 RCL    527  43 RCL    587  14  D
348  75  -     408  65  ×     468  03  03    528  26  26    588  53  (
349  43 RCL    409  43 RCL    469  65  ×     529  54  )     589  43 RCL
350  26  26    410  00  00    470  43 RCL    530  55  ÷     590  16  16
351  54  )     411  54  )     471  00  00    531  43 RCL    591  75  -
352  65  ×     412  22 INV    472  54  )     532  27  27    592  43 RCL
353  53  (     413  23 LNX    473  22 INV    533  85  +     593  19  19
354  43 RCL    414  65  ×     474  23 LNX    534  53  (     594  54  )
355  21  21    415  43 RCL    475  65  ×     535  43 RCL    595  94 +/-
356  65  ×     416  25  25    476  43 RCL    536  02  02    596  65  ×
357  43 RCL    417  65  ×     477  24  24    537  65  ×     597  53  (
358  25  25    418  53  (     478  65  ×     538  43 RCL    598  53  (
359  75  -     419  43 RCL    479  53  (     539  00  00    599  43 RCL
```

Figure 31. (continued)

```
600  01  01     660  53  (        10000.        00
601  65  ×      661  43 RCL   -.0003336519      01
602  43 RCL     662  21  21   -.2702561925      02
603  00  00     663  75  -    -.9739707142      03
604  54  )      664  43 RCL         -0.001      04
605  22 INV     665  22  22    2.976671851      05
606  23 LNX     666  54  )     .0011904762      06
607  65  ×      667  55  ÷           -0.91      07
608  43 RCL     668  43 RCL           0.91      08
609  25  25     669  29  29    .0451962483      09
610  65  ×      670  54  )    -.3336155586      10
611  53  (      671  54  )              0.      11
612  43 RCL     672  95  =              0.      12
613  22  22     673  92 RTN             0.      13
614  75  -      674  76 LBL             0.      14
615  43 RCL     675  15  E             90.      15
616  23  23     676  16 A'      440.471568      16
617  54  )      677  17 B'         267696.      17
618  55  ÷      678  18 C'             50.      18
619  43 RCL     679  19 D'     244.7064267      19
620  27  27     680  10 E'     148333.5925      20
621  85  +      681  11  A     .0006113431      21
622  53  (      682  61 GTO   -.2692561925      22
623  43 RCL     683  38 SIN   -.9729707142      23
624  02  02     684  76 LBL    .0012467543      24
625  65  ×      685  39 COS    .0013563243      25
626  43 RCL     686  43 RCL   -.0136027542      26
627  00  00     687  18  18     0.00000558      27
628  54  )      688  85  +    -.0000559628      28
629  22 INV     689  12  B     .0000051292      29
630  23 LNX     690  85  +
631  65  ×      691  13  C
632  43 RCL     692  85  +
633  26  26     693  14  D
634  65  ×      694  95  =
635  53  (      695  91 R/S
636  43 RCL     696  61 GTO
637  23  23     697  39 COS
638  75  -
639  43 RCL
640  21  21
641  54  )
642  55  ÷
643  43 RCL
644  28  29
645  85  +
646  53  (
647  53  (
648  43 RCL
649  03  03
650  65  ×
651  43 RCL
652  00  00
653  54  )
654  22 INV
655  23 LNX
656  65  ×
657  43 RCL
658  24  24
659  65  ×
```

Figure 31. (continued)

SEDIMENT INTERSTITIAL PHOSPHORUS CONCENTRATION

The following program calculates the sediment interstitial phosphorus concentration ($P_i(t)$) (equation (4.22)) in response to a change in the phosphorus inflow concentration ($P_o = P_{Lo}$, $t < 0$; $P_o = P_{L\infty}$, $t \geqslant 0$). The parameters X_1, X_2, X_3, X_4, X_5, X_6, X_7, α, θ and γ from the parameters program must be used.

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter time (days)	t	STO00	t
2. Run program		E	$P_i(t)$
3. Enter time (days)	t	STO00	t
4. Run program		R/S	$P_i(t)$

```
000  76 LBL    060  53  (     120  43 RCL    180  09  09    240  55  ÷
001  16 A'     061  43 RCL    121  09  09    181  54  )     241  43 RCL
002  53  (     062  05  05    122  54  )     182  55  ÷     242  08  08
003  43 RCL    063  65  ×     123  55  ÷     183  43 RCL    243  55  ÷
004  05  05    064  43 RCL    124  43 RCL    184  08  08    244  53  (
005  65  ×     065  09  09    125  08  08    185  55  ÷     245  43 RCL
006  43 RCL    066  55  ÷     126  55  ÷     186  53  (     246  03  03
007  09  09    067  53  (     127  53  (     187  43 RCL    247  75  -
008  55  ÷     068  43 RCL    128  43 RCL    188  02  02    248  43 RCL
009  53  (     069  03  03    129  01  01    189  75  -     249  04  04
010  43 RCL    070  75  -     130  75  -     190  43 RCL    250  54  )
011  01  01    071  43 RCL    131  43 RCL    191  04  04    251  55  ÷
012  75  -     072  04  04    132  04  04    192  54  )     252  53  (
013  43 RCL    073  54  )     133  54  )     193  55  ÷     253  43 RCL
014  04  04    074  55  ÷     134  55  ÷     194  53  (     254  03  03
015  54  )     075  53  (     135  53  (     195  43 RCL    255  75  -
016  55  ÷     076  43 RCL    136  43 RCL    196  02  02    256  43 RCL
017  53  (     077  03  03    137  01  01    197  75  -     257  10  10
018  43 RCL    078  75  -     138  75  -     198  43 RCL    258  54  )
019  01  01    079  43 RCL    139  43 RCL    199  10  10    259  95  =
020  75  -     080  10  10    140  10  10    200  54  )     260  42 STO
021  43 RCL    081  54  )     141  54  )     201  95  =     261  26  26
022  10  10    082  54  )     142  95  =     202  42 STO    262  92 RTN
023  54  )     083  95  =     143  42 STO    203  25  25    263  76 LBL
024  54  )     084  42 STO    144  24  24    204  76 LBL    264  38 SIN
025  95  =     085  23  23    145  92 RTN    205  11 A      265  53  (
026  42 STO    086  92 RTN    146  76 LBL    206  53  (     266  53  (
027  21  21    087  76 LBL    147  10 E'     207  53  (     267  43 RCL
028  92 RTN    088  19 D'     148  53  (     208  43 RCL    268  25  25
029  76 LBL    089  53  (     149  53  (     209  03  03    269  75  -
030  17 B'     090  53  (     150  43 RCL    210  75  -     270  43 RCL
031  53  (     091  43 RCL    151  02  02    211  43 RCL    271  24  24
032  43 RCL    092  01  01    152  75  -     212  04  04    272  54  )
033  05  05    093  75  -     153  43 RCL    213  54  )     273  65  ×
034  65  ×     094  43 RCL    154  04  04    214  65  ×     274  53  (
035  43 RCL    095  04  04    155  54  )     215  53  (     275  43 RCL
036  09  09    096  54  )     156  65  ×     216  43 RCL    276  22  22
037  55  ÷     097  65  ×     157  53  (     217  03  03    277  65  ×
038  53  (     098  53  (     158  43 RCL    218  75  -     278  43 RCL
039  43 RCL    099  43 RCL    159  02  02    219  43 RCL    279  26  26
040  02  02    100  01  01    160  75  -     220  07  07    280  75  -
041  75  -     101  75  -     161  43 RCL    221  54  )     281  43 RCL
042  43 RCL    102  43 RCL    162  07  07    222  65  ×     282  23  23
043  04  04    103  07  07    163  54  )     223  53  (     283  65  ×
044  54  )     104  54  )     164  65  ×     224  43 RCL    284  43 RCL
045  55  ÷     105  65  ×     165  53  (     225  03  03    285  25  25
046  53  (     106  53  (     166  43 RCL    226  75  -     286  54  )
047  43 RCL    107  43 RCL    167  02  02    227  43 RCL    287  75  -
048  02  02    108  01  01    168  75  -     228  10  10    288  53  (
049  75  -     109  75  -     169  43 RCL    229  54  )     289  43 RCL
050  43 RCL    110  43 RCL    170  10  10    230  75  -     290  25  25
051  10  10    111  10  10    171  54  )     231  43 RCL    291  75  -
052  54  )     112  54  )     172  75  -     232  05  05    292  43 RCL
053  54  )     113  75  -     173  43 RCL    233  65  ×     293  26  26
054  95  =     114  43 RCL    174  05  05    234  43 RCL    294  54  )
055  42 STO    115  05  05    175  65  ×     235  06  06    295  65  ×
056  22  22    116  65  ×     176  43 RCL    236  65  ×     296  53  (
057  92 RTN    117  43 RCL    177  06  06    237  43 RCL    297  43 RCL
058  76 LBL    118  06  06    178  65  ×     238  09  09    298  22  22
059  18 C'     119  65  ×     179  43 RCL    239  54  )     299  65  ×
```

Figure 32. Coding for TI-59 Programmable Calculator, Lake model equation $P_i(t)$ (4.22) solved by Trial Function Method.

```
300  43  RCL     360  29  29      420  65  ×       480  28  28      540  53  (
301  24  24      361  53  (       421  53  (       481  85  +       541  43  RCL
302  75  -       362  53  (       422  53  (       482  53  (       542  25  25
303  43  RCL     363  43  RCL     423  43  RCL     483  53  (       543  75  -
304  21  21      364  24  24      424  01  01      484  43  RCL     544  43  RCL
305  65  ×       365  75  -       425  65  ×       485  03  03      545  26  26
306  43  RCL     366  43  RCL     426  43  RCL     486  65  ×       546  54  )
307  25  25      367  26  26      427  00  00      487  43  RCL     547  55  ÷
308  54  )       368  54  )       428  54  )       488  00  00      548  43  RCL
309  54  )       369  65  ×       429  22  INV     489  54  )       549  27  27
310  95  =       370  53  (       430  23  LNX     490  22  INV     550  85  +
311  42  STO     371  43  RCL     431  65  ×       491  23  LNX     551  53  (
312  27  27      372  21  21      432  43  RCL     492  65  ×       552  43  RCL
313  53  (       373  65  ×       433  25  25      493  43  RCL     553  02  02
314  53  (       374  43  RCL     434  65  ×       494  24  24      554  65  ×
315  43  RCL     375  25  25      435  53  (       495  65  ×       555  43  RCL
316  26  26      376  75  -       436  43  RCL     496  53  (       556  00  00
317  75  -       377  43  RCL     437  22  22      497  43  RCL     557  54  )
318  43  RCL     378  22  22      438  65  ×       498  21  21      558  22  INV
319  25  25      379  65  ×       439  43  RCL     499  65  ×       559  23  LNX
320  54  )       380  43  RCL     440  26  26      500  43  RCL     560  65  ×
321  65  ×       381  24  24      441  75  -       501  25  25      561  43  RCL
322  53  (       382  54  )       442  43  RCL     502  75  -       562  26  26
323  43  RCL     383  75  -       443  23  23      503  43  RCL     563  65  ×
324  23  23      384  53  (       444  65  ×       504  24  24      564  53  (
325  65  ×       385  43  RCL     445  43  RCL     505  65  ×       565  43  RCL
326  43  RCL     386  24  24      446  25  25      506  43  RCL     566  26  26
327  24  24      387  75  -       447  54  )       507  22  22      567  75  -
328  75  -       388  43  RCL     448  55  ÷       508  54  )       568  43  RCL
329  43  RCL     389  25  25      449  43  RCL     509  55  ÷       569  24  24
330  21  21      390  54  )       450  27  27      510  43  RCL     570  54  )
331  65  ×       391  65  ×       451  85  +       511  29  29      571  55  ÷
332  43  RCL     392  53  (       452  53  (       512  54  )       572  43  RCL
333  26  26      393  43  RCL     453  43  RCL     513  54  )       573  28  28
334  54  )       394  21  21      454  02  02      514  95  =       574  85  +
335  75  -       395  65  ×       455  65  ×       515  92  RTN     575  53  (
336  53  (       396  43  RCL     456  43  RCL     516  76  LBL     576  53  (
337  43  RCL     397  26  26      457  00  00      517  13  C       577  43  RCL
338  26  26      398  75  -       458  54  )       518  53  (       578  03  03
339  75  -       399  43  RCL     459  22  INV     519  43  RCL     579  65  ×
340  43  RCL     400  23  23      460  23  LNX     520  17  17      580  43  RCL
341  24  24      401  65  ×       461  65  ×       521  75  -       581  00  00
342  54  )       402  43  RCL     462  43  RCL     522  43  RCL     582  54  )
343  65  ×       403  24  24      463  26  26      523  20  20      583  22  INV
344  53  (       404  54  )       464  65  ×       524  54  )       584  23  LNX
345  43  RCL     405  54  )       465  53  (       525  65  ×       585  65  ×
346  23  23      406  95  =       466  43  RCL     526  53  (       586  43  RCL
347  65  ×       407  42  STO     467  23  23      527  53  (       587  24  24
348  43  RCL     408  29  29      468  65  ×       528  43  RCL     588  65  ×
349  25  25      409  61  GTO     469  43  RCL     529  01  01      589  53  (
350  75  -       410  39  COS     470  24  24      530  65  ×       590  43  RCL
351  43  RCL     411  76  LBL     471  75  -       531  43  RCL     591  24  24
352  22  22      412  12  B       472  43  RCL     532  00  00      592  75  -
353  65  ×       413  53  (       473  21  21      533  54  )       593  43  RCL
354  43  RCL     414  43  RCL     474  65  ×       534  22  INV     594  25  25
355  26  26      415  16  16      475  43  RCL     535  23  LNX     595  54  )
356  54  )       416  75  -       476  26  26      536  65  ×       596  55  ÷
357  54  )       417  43  RCL     477  54  )       537  43  RCL     597  43  RCL
358  95  =       418  19  19      478  55  ÷       538  25  25      598  29  29
359  42  STO     419  54  )       479  43  RCL     539  65  ×       599  54  )
```

Figure 32. (continued)

```
600  54  )
601  95  =
602  92 RTN
603  76 LBL
604  14  D
605  53  (
606  43 RCL
607  15  15
608  75  -
609  43 RCL
610  18  18
611  54  )
612  94 +/-
613  65  ×
614  53  (
615  53  (
616  43 RCL
617  01  01
618  65  ×
619  43 RCL
620  00  00
621  54  )
622  22 INV
623  23 LNX
624  65  ×
625  43 RCL
626  25  25
627  65  ×
628  53  (
629  43 RCL
630  22  22
631  75  -
632  43 RCL
633  23  23
634  54  )
635  55  ÷
636  43 RCL
637  27  27
638  85  +
639  53  (
640  43 RCL
641  02  02
642  65  ×
643  43 RCL
644  00  00
645  54  )
646  22 INV
647  23 LNX
648  65  ×
649  43 RCL
650  26  26
651  65  ×
652  53  (
653  43 RCL
654  23  23
655  75  -
656  43 RCL
657  21  21
658  54  )
659  55  ÷
```

```
660  43 RCL
661  28  28
662  85  +
663  53  (
664  53  (
665  43 RCL
666  03  03
667  65  ×
668  43 RCL
669  00  00
670  54  )
671  22 INV
672  23 LNX
673  65  ×
674  43 RCL
675  24  24
676  65  ×
677  53  (
678  43 RCL
679  21  21
680  75  -
681  43 RCL
682  22  22
683  54  )
684  55  ÷
685  43 RCL
686  29  29
687  54  )
688  54  )
689  95  =
690  92 RTN
691  76 LBL
692  15  E
693  16 A'
694  17 B'
695  18 C'
696  19 D'
697  10 E'
698  11  A
699  61 GTO
700  38 SIN
701  76 LBL
702  39 COS
703  43 RCL
704  19  19
705  85  +
706  12  B
707  85  +
708  13  C
709  85  +
710  14  D
711  95  =
712  91 R/S
713  61 GTO
714  39 COS
```

```
          50.        00
-.0015807542        01
-.8107039075        02
-1.320771587        03
    -0.00186        04
 2.976671851        05
 .0022142857        06
       -0.91        07
        0.91        08
 .0451962483        09
-1.221196248        10
          0.        11
          0.        12
          0.        13
          0.        14
         90.        15
        450.        16
 144032.5089        17
         50.        18
 213.5533984        19
  80018.0605        20
 395.0243007        21
-.4051945258        22
 1.024391278        23
 .0370577801        24
 .1101025373        25
-.4533899797        26
 24.54349088        27
-101.1788179        28
 8.260729584        29
```

Figure 32. (continued)

SEDIMENT SOLID PHASE PHOSPHORUS CONCENTRATION

The following program calculates the sediment solid phase phosphorus concentration ($P_s(t)$) (equation (4.23)) in response to a change in the phosphorus inflow concentration ($P_o = P_{Lo}$, $t < 0$; $P_o = P_{L\infty}$, $t \geqslant 0$). The parameters X_1, X_2, X_3, X_4, X_5, X_6, X_7, α, β and γ from the parameters program must be used.

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter time (days)	t	STO00	t
2. Run program		E	$P_s(t)$
3. Enter time (days)	t	STO00	t
4. Run program		R/S	$P_s(t)$

```
000  76  LBL    060  54  )      120  53  (      180  43  RCL    240  43  RCL
001  16  A'     061  54  )      121  43  RCL    181  25   25    241  26   26
002  53  (      062  54  )      122  03   03    182  54  )      242  54  )
003  43  RCL    063  35  1/X    123  75  -      183  54  )      243  65  ×
004  01   01    064  65  ×      124  43  RCL    184  95  =      244  53  (
005  75  -      065  43  RCL    125  10   10    185  42  STO    245  43  RCL
006  43  RCL    066  06   06    126  54  )      186  27   27    246  21   21
007  04   04    067  95  =      127  54  )      187  53  (      247  65  ×
008  54  )      068  42  STO    128  54  )      188  53  (      248  43  RCL
009  95  =      069  24   24    129  35  1/X    189  43  RCL    249  25   25
010  42  STO    070  92  RTN    130  65  ×      190  26   26    250  75  -
011  21   21    071  76  LBL    131  43  RCL    191  75  -      251  43  RCL
012  92  RTN    072  10  E'     132  06   06    192  43  RCL    252  22   22
013  76  LBL    073  53  (      133  95  =      193  25   25    253  65  ×
014  17  B'     074  43  RCL    134  42  STO    194  54  )      254  43  RCL
015  53  (      075  02   02    135  26   26    195  65  ×      255  24   24
016  43  RCL    076  75  -      136  92  RTN    196  53  (      256  54  )
017  02   02    077  43  RCL    137  76  LBL    197  43  RCL    257  75  -
018  75  -      078  07   07    138  38  SIN    198  23   23    258  53  (
019  43  RCL    079  75  -      139  53  (      199  65  ×      259  43  RCL
020  04   04    080  53  (      140  53  (      200  43  RCL    260  24   24
021  54  )      081  43  RCL    141  43  RCL    201  24   24    261  75  -
022  95  =      082  09   09    142  25   25    202  75  -      262  43  RCL
023  42  STO    083  65  ×      143  75  -      203  43  RCL    263  25   25
024  22   22    084  43  RCL    144  43  RCL    204  21   21    264  54  )
025  92  RTN    085  10   10    145  24   24    205  65  ×      265  65  ×
026  76  LBL    086  55  ÷      146  54  )      206  43  RCL    266  53  (
027  18  C'     087  53  (      147  65  ×      207  26   26    267  43  RCL
028  53  (      088  43  RCL    148  53  (      208  54  )      268  21   21
029  43  RCL    089  02   02    149  43  RCL    209  75  -      269  65  ×
030  03   03    090  75  -      150  22   22    210  53  (      270  43  RCL
031  75  -      091  43  RCL    151  65  ×      211  43  RCL    271  26   26
032  43  RCL    092  10   10    152  43  RCL    212  26   26    272  75  -
033  04   04    093  54  )      153  26   26    213  75  -      273  43  RCL
034  54  )      094  54  )      154  75  -      214  43  RCL    274  23   23
035  95  =      095  54  )      155  43  RCL    215  24   24    275  65  ×
036  42  STO    096  35  1/X    156  23   23    216  54  )      276  43  RCL
037  23   23    097  65  ×      157  65  ×      217  65  ×      277  24   24
038  92  RTN    098  43  RCL    158  43  RCL    218  53  (      278  54  )
039  76  LBL    099  06   06    159  25   25    219  43  RCL    279  54  )
040  19  D'     100  95  =      160  54  )      220  23   23    280  95  =
041  53  (      101  42  STO    161  75  -      221  65  ×      281  42  STO
042  43  RCL    102  25   25    162  53  (      222  43  RCL    282  29   29
043  01   01    103  92  RTN    163  43  RCL    223  25   25    283  61  GTO
044  75  -      104  76  LBL    164  25   25    224  75  -      284  39  COS
045  43  RCL    105  11  A      165  75  -      225  43  RCL    285  76  LBL
046  07   07    106  53  (      166  43  RCL    226  22   22    286  12  B
047  75  -      107  43  RCL    167  26   26    227  65  ×      287  53  (
048  43  RCL    108  03   03    168  54  )      228  43  RCL    288  43  RCL
049  09   09    109  75  -      169  65  ×      229  26   26    289  17   17
050  65  ×      110  43  RCL    170  53  (      230  54  )      290  75  -
051  43  RCL    111  07   07    171  43  RCL    231  54  )      291  43  RCL
052  10   10    112  75  -      172  22   22    232  95  =      292  20   20
053  55  ÷      113  53  (      173  65  ×      233  42  STO    293  54  )
054  53  (      114  43  RCL    174  43  RCL    234  28   28    294  65  ×
055  43  RCL    115  09   09    175  24   24    235  53  (      295  53  (
056  01   01    116  65  ×      176  75  -      236  53  (      296  53  (
057  75  -      117  43  RCL    177  43  RCL    237  43  RCL    297  43  RCL
058  43  RCL    118  10   10    178  21   21    238  24   24    298  01   01
059  10   10    119  55  ÷      179  65  ×      239  75  -      299  65  ×
```

Figure 33. Coding for TI-59 Programmable Calculator, lake model equation $P_s(t)$ (4.23) solved by Trial Function Method.

```
300  43  RCL
301  00   00
302  54   )
303  22  INV
304  23  LNX
305  65   ×
306  43  RCL
307  25   25
308  65   ×
309  53   (
310  43  RCL
311  22   22
312  65   ×
313  43  RCL
314  26   26
315  75   -
316  43  RCL
317  23   23
318  65   ×
319  43  RCL
320  25   25
321  54   )
322  55   ÷
323  43  RCL
324  27   27
325  85   +
326  53   (
327  43  RCL
328  02   02
329  65   ×
330  43  RCL
331  00   00
332  54   )
333  22  INV
334  23  LNX
335  65   ×
336  43  RCL
337  26   26
338  65   ×
339  53   (
340  43  RCL
341  23   23
342  65   ×
343  43  RCL
344  24   24
345  75   -
346  43  RCL
347  21   21
348  65   ×
349  43  RCL
350  26   26
351  54   )
352  55   ÷
353  43  RCL
354  28   28
355  85   +
356  53   (
357  53   (
358  43  RCL
359  03   03
360  65   ×
361  43  RCL
362  00   00
363  54   )
364  22  INV
365  23  LNX
366  65   ×
367  43  RCL
368  24   24
369  65   ×
370  53   (
371  43  RCL
372  21   21
373  65   ×
374  43  RCL
375  25   25
376  75   -
377  43  RCL
378  24   24
379  65   ×
380  43  RCL
381  22   22
382  54   )
383  55   ÷
384  43  RCL
385  29   29
386  54   )
387  54   )
388  95   =
389  92  RTN
390  76  LBL
391  13   C
392  53   (
393  43  RCL
394  15   15
395  75   -
396  43  RCL
397  18   18
398  54   )
399  65   ×
400  53   (
401  53   (
402  43  RCL
403  01   01
404  65   ×
405  43  RCL
406  00   00
407  54   )
408  22  INV
409  23  LNX
410  65   ×
411  43  RCL
412  25   25
413  65   ×
414  53   (
415  43  RCL
416  25   25
417  75   -
418  43  RCL
419  26   26
420  54   )
421  55   ÷
422  43  RCL
423  27   27
424  85   +
425  53   (
426  43  RCL
427  02   02
428  65   ×
429  43  RCL
430  00   00
431  54   )
432  22  INV
433  23  LNX
434  65   ×
435  43  RCL
436  26   26
437  65   ×
438  53   (
439  43  RCL
440  26   26
441  75   -
442  43  RCL
443  24   24
444  54   )
445  55   ÷
446  43  RCL
447  28   28
448  85   +
449  53   (
450  53   (
451  43  RCL
452  03   03
453  65   ×
454  43  RCL
455  00   00
456  54   )
457  22  INV
458  23  LNX
459  65   ×
460  43  RCL
461  24   24
462  65   ×
463  53   (
464  43  RCL
465  24   24
466  75   -
467  43  RCL
468  25   25
469  54   )
470  55   ÷
471  43  RCL
472  29   29
473  54   )
474  54   )
475  95   =
476  92  RTN
477  76  LBL
478  14   D
479  53   (
480  43  RCL
481  16   16
482  75   -
483  43  RCL
484  19   19
485  54   )
486  94  +/-
487  65   ×
488  53   (
489  53   (
490  43  RCL
491  01   01
492  65   ×
493  43  RCL
494  00   00
495  54   )
496  22  INV
497  23  LNX
498  65   ×
499  43  RCL
500  25   25
501  65   ×
502  53   (
503  43  RCL
504  22   22
505  75   -
506  43  RCL
507  23   23
508  54   )
509  55   ÷
510  43  RCL
511  27   27
512  85   +
513  53   (
514  43  RCL
515  02   02
516  65   ×
517  43  RCL
518  00   00
519  54   )
520  22  INV
521  23  LNX
522  65   ×
523  43  RCL
524  26   26
525  65   ×
526  53   (
527  43  RCL
528  23   23
529  75   -
530  43  RCL
531  21   21
532  54   )
533  55   ÷
534  43  RCL
535  28   28
536  85   +
537  53   (
538  53   (
539  43  RCL
540  03   03
541  65   ×
542  43  RCL
543  00   00
544  54   )
545  22  INV
546  23  LNX
547  65   ×
548  43  RCL
549  24   24
550  65   ×
551  53   (
552  43  RCL
553  21   21
554  75   -
555  43  RCL
556  22   22
557  54   )
558  55   ÷
559  43  RCL
560  29   29
561  54   )
562  54   )
563  95   =
564  92  RTN
565  76  LBL
566  15   E
567  16   A'
568  17   B'
569  18   C'
570  19   D'
571  10   E'
572  11   A
573  61  GTO
574  38  SIN
575  76  LBL
576  39  COS
577  43  RCL
578  20   20
579  85   +
580  12   B
581  85   +
582  13   C
583  85   +
584  14   D
585  95   =
586  91  R/S
587  61  GTO
588  39  COS
```

Figure 33. (continued)

```
         500.        00
 -.0006305994        01
 -.2596287671        02
 -.9726868817        03
     -0.00175        04
  2.976671851        05
  .0020833333        06
        -0.91        07
         0.91        08
  .0451962483        09
 -.3211962483        10
           0.        11
           0.        12
           0.        13
           0.        14
          90.        15
         450.        16
      375000.        17
          50.        18
        245.3        19
      213000.        20
  .0011194006        21
 -.2578787671        22
 -.9709368817        23
  .0021822903        24
  .0023509698        25
 -.0245186247        26
  .0000166436        27
 -.0001735785        28
  .0000154494        29
```

Figure 33. (continued)

Equations (4.29, 30 and 31) obtained by approximation techniques.

LAKE WATER PHOSPHORUS CONCENTRATION

Approximate Equation

The following program calculates the lake water phosphorus concentration ($P_L(t)$) (equation (4.31)) in response to a change in the phosphorus inflow concentration ($P_o = P_{Lo}$, $t < 0$; $P_o = P_{L\infty}$, $t \geqslant 0$).

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter Parameters	P_{io}	STO08	P_{io}
	P_{so}	STO09	P_{so}
	P_o	STO10	P_o
	K_1	STO11	K_1
	K_2	STO12	K_2
	K_3	STO13	K_3
	Z	STO14	Z
	Dr	STO15	Dr
	T	STO16	T
	$\mathcal{E}$	STO17	$\mathcal{E}$
2. Enter time (days)	t	STO00	t
3. Run program		E	$P_L(t)$
4. Enter time (days)	t	STO00	t
5. Run program		R/S	$P_L(t)$

```
000  76 LBL    060  14  14    120  55  ÷     180  43 RCL    240  00  00
001  11  A     061  33 X²     121  43 RCL    181  00  00    241  54  )
002  53  (     062  65  ×     122  18  18    182  54  )     242  22 INV
003  43 RCL    063  43 RCL    123  94 +/-    183  22 INV    243  23 LNX
004  14  14    064  10  10    124  54  )     184  23 LNX    244  65  ×
005  65  ×     065  65  ×     125  42 STO    185  65  ×     245  53  (
006  43 RCL    066  43 RCL    126  06  06    186  53  (     246  53  (
007  15  15    067  12  12    127  53  (     187  53  (     247  53  (
008  85  +     068  55  ÷     128  53  (     188  53  (     248  53  (
009  43 RCL    069  43 RCL    129  53  (     189  53  (     249  43 RCL
010  16  16    070  18  18    130  43 RCL    190  43 RCL    250  05  05
011  65  ×     071  94 +/-    131  05  05    191  05  05    251  85  +
012  43 RCL    072  54  )     132  75  -     192  85  +     252  43 RCL
013  15  15    073  42 STO    133  43 RCL    193  43 RCL    253  01  01
014  65  ×     074  03  03    134  01  01    194  01  01    254  54  )
015  53  (     075  53  (     135  54  )     195  54  )     255  55  ÷
016  43 RCL    076  43 RCL    136  33 X²     196  94 +/-    256  02  2
017  17  17    077  13  13    137  75  -     197  55  ÷     257  94 +/-
018  65  ×     078  55  ÷     138  04  4     198  02  2     258  75  -
019  43 RCL    079  43 RCL    139  65  ×     199  85  +     259  43 RCL
020  11  11    080  17  17    140  53  (     200  43 RCL    260  07  07
021  85  +     081  54  )     141  43 RCL    201  07  07    261  54  )
022  43 RCL    082  42 STO    142  02  02    202  54  )     262  65  ×
023  12  12    083  04  04    143  65  ×     203  65  ×     263  43 RCL
024  65  ×     084  53  (     144  43 RCL    204  43 RCL    264  08  08
025  43 RCL    085  53  (     145  04  04    205  08  08    265  85  +
026  14  14    086  43 RCL    146  75  -     206  85  +     266  43 RCL
027  54  )     087  11  11    147  43 RCL    207  43 RCL    267  04  04
028  54  )     088  33 X²     148  01  01    208  09  09    268  65  ×
029  42 STO    089  65  ×     149  65  ×     209  65  ×     269  43 RCL
030  18  18    090  43 RCL    150  43 RCL    210  43 RCL    270  09  09
031  43 RCL    091  17  17    151  05  05    211  04  04    271  54  )
032  13  13    092  65  ×     152  54  )     212  54  )     272  55  ÷
033  94 +/-    093  43 RCL    153  54  )     213  55  ÷     273  43 RCL
034  42 STO    094  16  16    154  34 ΓX     214  43 RCL    274  07  07
035  01  01    095  55  ÷     155  55  ÷     215  07  07    275  94 +/-
036  53  (     096  43 RCL    156  02  2     216  55  ÷     276  55  ÷
037  43 RCL    097  18  18    157  54  )     217  02  2     277  02  2
038  17  17    098  94 +/-    158  42 STO    218  54  )     278  54  )
039  65  ×     099  54  )     159  07  07    219  92 RTN    279  92 RTN
040  43 RCL    100  85  +     160  92 RTN    220  76 LBL    280  76 LBL
041  11  11    101  53  (     161  76 LBL    221  17 B'     281  13  C
042  65  ×     102  43 RCL    162  12  B     222  53  (     282  53  (
043  43 RCL    103  11  11    163  53  (     223  53  (     283  53  (
044  12  12    104  55  ÷     164  53  (     224  53  (     284  53  (
045  65  ×     105  43 RCL    165  53  (     225  43 RCL    285  53  (
046  43 RCL    106  15  15    166  43 RCL    226  05  05    286  53  (
047  14  14    107  54  )     167  05  05    227  75  -     287  43 RCL
048  65  ×     108  54  )     168  75  -     228  43 RCL    288  05  05
049  43 RCL    109  42 STO    169  43 RCL    229  01  01    289  75  -
050  16  16    110  05  05    170  01  01    230  54  )     290  43 RCL
051  55  ÷     111  53  (     171  54  )     231  94 +/-    291  01  01
052  43 RCL    112  43 RCL    172  94 +/-    232  55  ÷     292  54  )
053  18  18    113  10  10    173  55  ÷     233  02  2     293  55  ÷
054  94 +/-    114  65  ×     174  02  2     234  75  -     294  02  2
055  54  )     115  43 RCL    175  85  +     235  43 RCL    295  94 +/-
056  42 STO    116  11  11    176  43 RCL    236  07  07    296  75  -
057  02  02    117  65  ×     177  07  07    237  54  )     297  43 RCL
058  53  (     118  43 RCL    178  54  )     238  65  ×     298  07  07
059  43 RCL    119  14  14    179  65  ×     239  43 RCL    299  54  )
```

Figure 34. Coding for TI-59 Programmable Calculator, lake model equation $P_L(t)$ solved by Approximation techniques.

```
300  65  ×      360  92  RTN    420  07  07       480  54  )
301  43  RCL    361  76  LBL    421  55  ÷        481  95  =
302  00  00     362  18  C'     422  02  2        482  91  R/S
303  54  )      363  53  (      423  55  ÷        483  61  GTO
304  22  INV    364  53  (      424  53  (        484  10  E'
305  23  LNX    365  53  (      425  53  (           1000.          00
306  94  +/-    366  53  (      426  43  RCL         -0.001         01
307  85  +      367  53  (      427  05  05      -.4188091954       02
308  01  1      368  43  RCL    428  75  -       -46.29693487       03
309  54  )      369  05  05     429  43  RCL     .0011904762        04
310  65  ×      370  75  -      430  01  01      .7818678161        05
311  53  (      371  43  RCL    431  54  )       -14.16427203       06
312  53  (      372  01  01     432  94  +/-     .3910718846        07
313  53  (      373  54  )      433  55  ÷             440.5        08
314  43  RCL    374  94  +/-    434  02  2           267690.        09
315  05  05     375  55  ÷      435  85  +               50.        10
316  85  +      376  02  2      436  43  RCL           0.091        11
317  43  RCL    377  85  +      437  07  07            0.176        12
318  01  01     378  43  RCL    438  54  )             0.001        13
319  54  )      379  07  07     439  54  )              1.69        14
320  55  ÷      380  54  )      440  92  RTN             0.1        15
321  02  2      381  65  ×      441  76  LBL             10.        16
322  94  +/-    382  43  RCL    442  15  E              0.34        17
323  75  -      383  00  00     443  11  A           0.54288        18
324  43  RCL    384  54  )      444  76  LBL              0.        19
325  07  07     385  22  INV    445  10  E'               0.        20
326  54  )      386  23  LNX    446  53  (                0.        21
327  65  ×      387  94  +/-    447  12  B                0.        22
328  43  RCL    388  85  +      448  85  +                0.        23
329  06  06     389  01  1      449  17  B'               0.        24
330  85  +      390  54  )      450  85  +                0.        25
331  43  RCL    391  65  ×      451  13  C                0.        26
332  04  04     392  53  (      452  85  +                0.        27
333  65  ×      393  53  (      453  18  C'               0.        28
334  43  RCL    394  53  (      454  54  )                0.        29
335  03  03     395  43  RCL    455  65  ×                0.        30
336  54  )      396  05  05     456  53  (                0.        31
337  55  ÷      397  85  +      457  43  RCL              0.        32
338  43  RCL    398  43  RCL    458  02  02               0.        33
339  07  07     399  01  01     459  65  ×                0.        34
340  55  ÷      400  54  )      460  43  RCL              0.        35
341  02  2      401  94  +/-    461  15  15               0.        36
342  55  ÷      402  55  ÷      462  55  ÷                0.        37
343  53  (      403  02  2      463  43  RCL              0.        38
344  53  (      404  85  +      464  12  12               0.        39
345  43  RCL    405  43  RCL    465  55  ÷
346  05  05     406  07  07     466  43  RCL
347  75  -      407  54  )      467  14  14
348  43  RCL    408  65  ×      468  94  +/-
349  01  01     409  43  RCL    469  54  )
350  54  )      410  06  06     470  75  -
351  94  +/-    411  85  +      471  53  (
352  55  ÷      412  43  RCL    472  43  RCL
353  02  2      413  04  04     473  06  06
354  75  -      414  65  ×      474  65  ×
355  43  RCL    415  43  RCL    475  43  RCL
356  07  07     416  03  03     476  15  15
357  54  )      417  54  )      477  55  ÷
358  54  )      418  55  ÷      478  43  RCL
359  94  +/-    419  43  RCL    479  11  11
```

Figure 34. (continued)

SEDIMENT INTERSTITIAL PHOSPHORUS CONCENTRATION

Approximate Equation

The following program calculates the sediment interstitial phosphorus concentration ($P_i(t)$) (equation (4.29)) in response to a change in phosphorus inflow concentration ($P_o = P_{Lo}$, $t < 0$; $P_o = P_{L\infty}$, $t \geqslant 0$).

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter Parameters	P_{io}	STO08	P_{io}
	P_{so}	STO09	P_{so}
	P_o	STO10	P_o
	K_1	STO11	K_1
	K_2	STO12	K_2
	K_3	STO13	K_3
	Z	STO14	Z
	Dr	STO15	Dr
	T	STO16	T
	$\mathcal{E}$	STO17	$\mathcal{E}$
2. Enter time (days)	t	STO00	t
3. Run program		E	$P_i(t)$
4. Enter time (days)	t	STO00	t
5. Run program		R/S	$P_i(t)$

```
000  76 LBL     060  14  14     120  55  ÷      180  43 RCL     240  00  00
001  11  A      061  33 X²      121  43 RCL     181  00  00     241  54  )
002  53  (      062  65  ×      122  18  18     182  54  )      242  22 INV
003  43 RCL     063  43 RCL     123  94 +/-     183  22 INV     243  23 LNX
004  14  14     064  10  10     124  54  )      184  23 LNX     244  65  ×
005  65  ×      065  65  ×      125  42 STO     185  65  ×      245  53  (
006  43 RCL     066  43 RCL     126  06  06     186  53  (      246  53  (
007  15  15     067  12  12     127  53  (      187  53  (      247  53  (
008  85  +      068  55  ÷      128  53  (      188  53  (      248  53  (
009  43 RCL     069  43 RCL     129  53  (      189  53  (      249  43 RCL
010  16  16     070  18  18     130  43 RCL     190  43 RCL     250  05  05
011  65  ×      071  94 +/-     131  05  05     191  05  05     251  85  +
012  43 RCL     072  54  )      132  75  -      192  85  +      252  43 RCL
013  15  15     073  42 STO     133  43 RCL     193  43 RCL     253  01  01
014  65  ×      074  03  03     134  01  01     194  01  01     254  54  )
015  53  (      075  53  (      135  54  )      195  54  )      255  55  ÷
016  43 RCL     076  43 RCL     136  33 X²      196  94 +/-     256  02  2
017  17  17     077  13  13     137  75  -      197  55  ÷      257  94 +/-
018  65  ×      078  55  ÷      138  04  4      198  02  2      258  75  -
019  43 RCL     079  43 RCL     139  65  ×      199  85  +      259  43 RCL
020  11  11     080  17  17     140  53  (      200  43 RCL     260  07  07
021  85  +      081  54  )      141  43 RCL     201  07  07     261  54  )
022  43 RCL     082  42 STO     142  02  02     202  54  )      262  65  ×
023  12  12     083  04  04     143  65  ×      203  65  ×      263  43 RCL
024  65  ×      084  53  (      144  43 RCL     204  43 RCL     264  08  08
025  43 RCL     085  53  (      145  04  04     205  08  08     265  85  +
026  14  14     086  43 RCL     146  75  -      206  85  +      266  43 RCL
027  54  )      087  11  11     147  43 RCL     207  43 RCL     267  04  04
028  54  )      088  33 X²      148  01  01     208  09  09     268  65  ×
029  42 STO     089  65  ×      149  65  ×      209  65  ×      269  43 RCL
030  18  18     090  43 RCL     150  43 RCL     210  43 RCL     270  09  09
031  43 RCL     091  17  17     151  05  05     211  04  04     271  54  )
032  13  13     092  65  ×      152  54  )      212  54  )      272  55  ÷
033  94 +/-     093  43 RCL     153  54  )      213  55  ÷      273  43 RCL
034  42 STO     094  16  16     154  34 ΓX      214  43 RCL     274  07  07
035  01  01     095  55  ÷      155  55  ÷      215  07  07     275  94 +/-
036  53  (      096  43 RCL     156  02  2      216  55  ÷      276  55  ÷
037  43 RCL     097  18  18     157  54  )      217  02  2      277  02  2
038  17  17     098  94 +/-     158  42 STO     218  54  )      278  54  )
039  65  ×      099  54  )      159  07  07     219  92 RTN     279  92 RTN
040  43 RCL     100  85  +      160  92 RTN     220  76 LBL     280  76 LBL
041  11  11     101  53  (      161  76 LBL     221  17  B'     281  13  C
042  65  ×      102  43 RCL     162  12  B      222  53  (      282  53  (
043  43 RCL     103  11  11     163  53  (      223  53  (      283  53  (
044  12  12     104  55  ÷      164  53  (      224  53  (      284  53  (
045  65  ×      105  43 RCL     165  53  (      225  43 RCL     285  53  (
046  43 RCL     106  15  15     166  43 RCL     226  05  05     286  53  (
047  14  14     107  54  )      167  05  05     227  75  -      287  43 RCL
048  65  ×      108  54  )      168  75  -      228  43 RCL     288  05  05
049  43 RCL     109  42 STO     169  43 RCL     229  01  01     289  75  -
050  16  16     110  05  05     170  01  01     230  54  )      290  43 RCL
051  55  ÷      111  53  (      171  54  )      231  94 +/-     291  01  01
052  43 RCL     112  43 RCL     172  94 +/-     232  55  ÷      292  54  )
053  18  18     113  10  10     173  55  ÷      233  02  2      293  55  ÷
054  94 +/-     114  65  ×      174  02  2      234  75  -      294  02  2
055  54  )      115  43 RCL     175  85  +      235  43 RCL     295  94 +/-
056  42 STO     116  11  11     176  43 RCL     236  07  07     296  75  -
057  02  02     117  65  ×      177  07  07     237  54  )      297  43 RCL
058  53  (      118  43 RCL     178  54  )      238  65  ×      298  07  07
059  43 RCL     119  14  14     179  65  ×      239  43 RCL     299  54  )
```

Figure 35. Coding for TI-59 Programmable Calculator, lake model equation $P_i(t)$ solved by Approximation techniques.

```
300  65   ×      360  92  RTN          420  07  07
301  43  RCL     361  76  LBL          421  55   ÷
302  00   00     362  18  C'           422  02   2
303  54   )      363  53   (           423  55   ÷
304  22  INV     364  53   (           424  53   (
305  23  LNX     365  53   (           425  53   (
306  94  +/-     366  53   (           426  43  RCL
307  85   +      367  53   (           427  05   05
308  01   1      368  43  RCL          428  75   -
309  54   )      369  05   05          429  43  RCL
310  65   ×      370  75   -           430  01   01
311  53   (      371  43  RCL          431  54   )
312  53   (      372  01   01          432  94  +/-
313  53   (      373  54   )           433  55   ÷
314  43  RCL     374  94  +/-          434  02   2
315  05   05     375  55   ÷           435  85   +
316  85   +      376  02   2           436  43  RCL
317  43  RCL     377  85   +           437  07   07
318  01   01     378  43  RCL          438  54   )
319  54   )      379  07   07          439  54   )
320  55   ÷      380  54   )           440  92  RTN
321  02   2      381  65   ×           441  76  LBL
322  94  +/-     382  43  RCL          442  15   E
323  75   -      383  00   00          443  11   A
324  43  RCL     384  54   )           444  76  LBL
325  07   07     385  22  INV          445  10  E'
326  54   )      386  23  LNX          446  53   (
327  65   ×      387  94  +/-          447  12   B
328  43  RCL     388  85   +           448  85   +
329  06   06     389  01   1           449  17  B'
330  85   +      390  54   )           450  85   +
331  43  RCL     391  65   ×           451  13   C
332  04   04     392  53   (           452  85   +
333  65   ×      393  53   (           453  18  C'
334  43  RCL     394  53   (           454  54   )
335  03   03     395  43  RCL          455  91  R/S
336  54   )      396  05   05          456  61  GTO
337  55   ÷      397  85   +           457  10  E'
338  43  RCL     398  43  RCL                   0.           00
339  07   07     399  01   01              -0.001           01
340  55   ÷      400  54   )          -.4138091954           02
341  02   2      401  94  +/-          -46.29633487           03
342  55   ÷      402  55   ÷          .0011904762           04
343  53   (      403  02   2           .7818678161           05
344  53   (      404  85   +          -14.16427203           06
345  43  RCL     405  43  RCL          .3910718846           07
346  05   05     406  07   07                 450.           08
347  75   -      407  54   )              267690.           09
348  43  RCL     408  65   ×                  50.           10
349  01   01     409  43  RCL               0.091           11
350  54   )      410  06   06               0.176           12
351  94  +/-     411  85   +                0.001           13
352  55   ÷      412  43  RCL                1.69           14
353  02   2      413  04   04                 0.1           15
354  75   -      414  65   ×                  10.           16
355  43  RCL     415  43  RCL                0.84           17
356  07   07     416  03   03             0.54288           18
357  54   )      417  54   )
358  54   )      418  55   ÷
359  94  +/-     419  43  RCL
```

Figure 35. (continued)

SEDIMENT SOLID PHASE PHOSPHORUS CONCENTRATION

Approximate Equation

This program calculates the sediment solid phase phosphorus concentration ($P_s(t)$) (equation (4.30)) in response to a change in phosphorus inflow concentration ($P_o = P_{Lo}$, $t < 0$; $P_o = P_{L\infty}$, $t \geq 0$).

USER INSTRUCTIONS

Procedure	Enter	Press	Display
1. Enter Parameters	A	STO10	A
	K_1	STO11	K_1
	K_2	STO12	K_2
	K_3	STO13	K_3
	Q	STO14	Q
	V_L	STO15	V_L
	V_s	STO16	V_s
	ε	STO17	ε
	P_o	STO18	P_o
	P_{io}	STO19	P_{io}
	P_{so}	STO20	P_{so}
2. Enter time (days)	t	STO00	t
3. Run program		E	$P_i(t)$
4. Enter time (days)	t	STO00	t
5. Run program		R/S	$P_i(t)$

```
000  76 LBL    060  06  06    120  07  07    180  07  07    240  00  0
001  11  A     061  53  (     121  54  )     181  85  +     241  54  )
002  43 RCL    062  53  (     122  33 X²     182  43 RCL    242  22 INV
003  13  13    063  43 RCL    123  75  -     183  01  01    243  23 LNX
004  94 +/-    064  14  14    124  04  4     184  65  ×     244  65  ×
005  42 STO    065  85  +     125  65  ×     185  43 RCL    245  53  (
006  01  01    066  43 RCL    126  53  (     186  07  07    246  53  (
007  53  (     067  10  10    127  43 RCL    187  54  )     247  53  (
008  43 RCL    068  65  ×     128  02  02    188  55  ÷     248  53  (
009  12  12    069  43 RCL    129  65  ×     189  02  2     249  43 RCL
010  65  ×     070  11  11    130  43 RCL    190  55  ÷     250  05  05
011  43 RCL    071  85  +     131  06  06    191  43 RCL    251  85  +
012  15  15    072  43 RCL    132  65  ×     192  07  07    252  43 RCL
013  55  ÷     073  12  12    133  43 RCL    193  54  )     253  01  01
014  43 RCL    074  65  ×     134  03  03    194  42 STO    254  54  )
015  16  16    075  43 RCL    135  55  ÷     195  21  21    255  55  ÷
016  54  )     076  15  15    136  43 RCL    196  53  (     256  02  2
017  42 STO    077  54  )     137  07  07    197  53  (     257  94 +/-
018  02  02    078  94 +/-    138  75  -     198  43 RCL    258  75  -
019  53  (     079  55  ÷     139  43 RCL    199  05  05    259  43 RCL
020  43 RCL    080  43 RCL    140  01  01    200  65  ×     260  07  07
021  13  13    081  15  15    141  65  ×     201  43 RCL    261  54  )
022  55  ÷     082  54  )     142  53  (     202  06  06    262  65  ×
023  43 RCL    083  42 STO    143  53  (     203  75  -     263  43 RCL
024  17  17    084  07  07    144  43 RCL    204  43 RCL    264  08  08
025  54  )     085  53  (     145  05  05    205  04  04    265  85  +
026  42 STO    086  43 RCL    146  65  ×     206  65  ×     266  43 RCL
027  03  03    087  18  18    147  43 RCL    207  43 RCL    267  04  04
028  53  (     088  65  ×     148  06  06    208  07  07    268  65  ×
029  43 RCL    089  43 RCL    149  75  -     209  75  -     269  43 RCL
030  10  10    090  14  14    150  43 RCL    210  43 RCL    270  09  09
031  94 +/-    091  55  ÷     151  04  04    211  01  01    271  54  )
032  65  ×     092  43 RCL    152  65  ×     212  65  ×     272  55  ÷
033  43 RCL    093  15  15    153  43 RCL    213  43 RCL    273  43 RCL
034  11  11    094  54  )     154  07  07    214  07  07    274  07  07
035  55  ÷     095  42 STO    155  54  )     215  54  )     275  94 +/-
036  43 RCL    096  08  08    156  55  ÷     216  55  ÷     276  55  ÷
037  16  16    097  53  (     157  43 RCL    217  02  2     277  02  2
038  54  )     098  53  (     158  07  07    218  55  ÷     278  54  )
039  42 STO    099  53  (     159  54  )     219  43 RCL    279  92 RTN
040  04  04    100  43 RCL    160  54  )     220  07  07    280  76 LBL
041  43 RCL    101  05  05    161  54  )     221  54  )     281  13  C
042  04  04    102  65  ×     162  34 √X     222  42 STO    282  53  (
043  94 +/-    103  43 RCL    163  55  ÷     223  22  22    283  53  (
044  42 STO    104  06  06    164  02  2     224  92 RTN    284  53  (
045  05  05    105  75  -     165  54  )     225  76 LBL    285  53  (
046  53  (     106  43 RCL    166  42 STO    226  10 E'     286  53  (
047  43 RCL    107  04  04    167  23  23    227  53  (     287  43 RCL
048  17  17    108  65  ×     168  53  (     228  53  (     288  05  05
049  65  ×     109  43 RCL    169  53  (     229  53  (     289  75  -
050  43 RCL    110  07  07    170  43 RCL    230  43 RCL    290  43 RCL
051  10  10    111  75  -     171  05  05    231  22  22    291  01  01
052  65  ×     112  43 RCL    172  65  ×     232  75  -     292  54  )
053  43 RCL    113  01  01    173  43 RCL    233  43 RCL    293  55  ÷
054  11  11    114  65  ×     174  06  06    234  23  23    294  02  2
055  55  ÷     115  43 RCL    175  75  -     235  54  )     295  94 +/-
056  43 RCL    116  07  07    176  43 RCL    236  65  ×     296  75  -
057  15  15    117  54  )     177  04  04    237  43 RCL    297  43 RCL
058  54  )     118  55  ÷     178  65  ×     238  00  00    298  07  07
059  42 STO    119  43 RCL    179  43 RCL    239  94 +/-    299  54  )
```

Figure 36. Coding for TI-59 Programmable Calculator, lake model equation $P_s(t)$ solved by Approximation techniques.

```
300  21  21
301  75  -
302  43 RCL
303  23  23
304  54  )
305  75  -
306  53  (
307  43 RCL
308  19  19
309  65  ×
310  43 RCL
311  02  02
312  65  ×
313  43 RCL
314  06  06
315  55  ÷
316  43 RCL
317  07  07
318  54  )
319  54  )
320  55  ÷
321  02  2
322  55  ÷
323  43 RCL
324  23  23
325  54  )
326  75  -
327  53  (
328  53  (
329  53  (
330  43 RCL
331  22  22
332  85  +
333  43 RCL
334  23  23
335  54  )
336  65  ×
337  43 RCL
338  00  00
339  94 +/-
340  54  )
341  22 INV
342  23 LNX
343  94 +/-
344  85  +
345  01  1
346  54  )
347  65  ×
348  53  (
349  43 RCL
350  02  02
351  65  ×
352  43 RCL
353  08  08
354  65  ×
355  53  (
356  43 RCL
357  06  06
358  65  ×
359  43 RCL
```

```
360  05  05
361  55  ÷
362  43 RCL
363  07  07
364  33 X²
365  75  -
366  01  1
367  54  )
368  65  ×
369  53  (
370  43 RCL
371  21  21
372  75  -
373  43 RCL
374  23  23
375  54  )
376  55  ÷
377  02  2
378  55  ÷
379  43 RCL
380  23  23
381  55  ÷
382  53  (
383  43 RCL
384  22  22
385  85  +
386  43 RCL
387  23  23
388  54  )
389  55  ÷
390  43 RCL
391  07  07
392  54  )
393  54  )
394  85  +
395  53  (
396  53  (
397  53  (
398  43 RCL
399  22  22
400  75  -
401  43 RCL
402  23  23
403  54  )
404  65  ×
405  43 RCL
406  00  00
407  94 +/-
408  54  )
409  22 INV
410  23 LNX
411  94 +/-
412  85  +
413  01  1
414  54  )
415  65  ×
416  53  (
417  43 RCL
418  02  02
419  65  ×
```

```
      420  43 RCL
      421  08  08
      422  65  ×
      423  53  (
      424  43 RCL
      425  06  06
      426  65  ×
      427  43 RCL
      428  05  05
      429  55  ÷
      430  43 RCL
      431  07  07
      432  33 X²
      433  75  -
      434  01  1
      435  54  )
      436  65  ×
      437  53  (
      438  43 RCL
      439  21  21
      440  85  +
      441  43 RCL
      442  23  23
      443  54  )
      444  55  ÷
      445  02  2
      446  55  ÷
      447  43 RCL
      448  23  23
      449  55  ÷
      450  53  (
      451  43 RCL
      452  22  22
      453  75  -
      454  43 RCL
      455  23  23
      456  54  )
      457  55  ÷
      458  43 RCL
      459  07  07
      460  54  )
      461  54  )
      462  54  )
      463  54  )
      464  92 RTN
      465  76 LBL
      466  15  E
      467  11  A
      468  10 E'
      469  91 R/S
      470  61 GTO
      471  10 E'
              0.        00
         -0.001         01
    2.976671851         02
    .0011904762         03
          -0.91         04
           0.91         05
    .0451962483         06
   -.3422188506         07
```

```
5.620689655    08
         0.    09
    257200.    10
      0.091    11
      0.176    12
      0.001    13
     48900.    14
    435000.    15
     25720.    16
       0.84    17
        50.    18
      440.5    19
    267690.    20
.3944089415    21
.3954089415    22
.3950017946    23
```

Figure 36. (continued)